The Elasticity of Space and the Nature of Irrationals

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"Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. The basic elements are logic and intuition, analysis and construction, generality and individuality."

(Courant and Robbins, "What is Mathematics?")

The above definition involves six antithetical elements expressed by the mind through will, reason, and artistic sensibility. What does a mathematician do with these elements? He explores numericity. Numericity is the ability to form precise groupings or collections of objects in various relationships. Numericity is sometimes called "mapping", and groupings or collections are sometimes called "sets". Mathematics arose as a technology for counting and measuring things – a technology of theoretical or applied numericity. Its basic subject matter is to discover and/or record the precise possible relationships among various creations when subjected to certain defined rules. Such creations may be of any kind, and the rules for analysis and construction (postulates) and logic can be of any kind.

Mathematicians generally believe that natural numbers, integers, and rational numbers are infinitely many, but still countable. They are also **equally** infinite – these sets are of equal size and we can map them in one-to-one correspondence. On the other hand, they commonly assert that irrational numbers – the additional numbers that form a continuous set of real numbers – are infinitely more numerous than the natural numbers, integers, and rational numbers and therefore somehow fundamentally uncountable. Most of mathematics involves working with real numbers or their equivalents (such as continuous lines). When mathematics becomes primarily populated by mathematical objects that **in principle** can not be counted, then I start to wonder whether it is turning into "mythematics".

I believe the problem with the set of real numbers arises from the way in which we understand the notion of infinity. Infinity does not mean that there is immeasurably "more" of something. "Infinite" means "not bounded", "not defined". Infinite space does not mean that space goes on and on forever. It simply means that nobody has bothered to probe its limits or to place some limits on it. It means the size of the space is indefinite. This, of course, may include a sense of largeness – but not necessarily. To say that there are infinitely many natural numbers (counting numbers) simply means that there are as many as you like. You are free to choose how many you would like to use. The upper boundary of the set of natural numbers is undefined. **You decide** if and where to terminate a counting process. A variable is another form of infinity. In ordinary algebra we are free to assign any number of possible values to a variable. An algebraic expression (e.g., $y = x + 3$) defines a relationship that may exist among constants (3), variables (*x*), and operators (+) according to certain assumed rules (postulates). Ordinarily the variables express the property of "infinity" in algebraic expressions. A special kind of infinity, represented by the ∞ sign, means that we consider the whole set of natural numbers (or an equivalent set) all at once. However, if we use this notion of "infinitely many" as a number in equations, the rules break down (e.g., $\infty + 5 = \infty$?! $\infty - \infty = 5$???)

The freedom to make decisions in the steps of a process is the essence of Observer Physics (OP) and Observer Mathematics (OM). From this viewpoint natural numbers and irrational numbers are equally "uncountable". They are both open-ended sets that leave one or more boundaries up to the "counter" to define. The practical consequence of this is that mathematics (and its companion, physics) empower the individual to make decisions about how to interpret the mental and physical world.

Non-algebraic irrational numbers have a peculiar property that differentiates them from rational numbers. The value of such a number can not be precisely defined. It is in this sense alone that such a set can be considered to be "more" infinite than natural or rational numbers. Mathematicians generally believe that they can not precisely define the value of any particular non-algebraic irrational number. This is a problem of viewpoint. It is like saying you can not count rational numbers in "numerical" order ($b > a$) in the same way that you can count the natural numbers or integers.

1/1, 1/2, 1/3, 1/4, 1/5, ...
2/1, 2/2, 2/3, 2/4, 2/5, ... = 1/1, 1/2, 2/1, 3/1, 2/2, 1/3, 1/4, 2/3, 3/2, ...
3/1, 3/2, 3/3, 3/4, 3/5, ...
. . . .

No matter how you try to count rational numbers, you can not make them line up in numerical order. However, you **can** lay them out in orderly arrays. And Cantor showed that you also **can** count rational numbers in an orderly grid by proceeding diagonally from the most clearly defined corner. In this way we can place the set of rational numbers into a one-to-one correspondence with the natural numbers. This demonstration discovered by Cantor also hints at the elasticity of space: all the rational numbers when laid out in a square grid equal all the natural numbers when laid out in a single row. The same thing turns out to be true for continuous sets: lines and planes (and n-dimensional spaces) all contain the same number of points.

Cantor used another diagonal method to demonstrate that you can always add new irrational numbers to any supposedly complete list. He set up a hypothetical list of **all** the infinite decimals between 0 and 1 (a complete set of real numbers), and then "diagonalized" it to create a new number by changing the first digit of the first number on the list, the second digit of the second number on the list, the third digit of the third number on the list, and so on. This apparently generates a new number that is not on his so-called complete list. Unfortunately, such a "demonstration proof" does not prove anything at all, because we can also do this with any supposedly complete list of natural numbers. Given a complete list that goes 1, 2, 3, 4, . . . n, we can always come up with (n + 1), a new number that is apparently not on the list. But, when we look closely at the new number, it turns out to be a perfectly good number on the list. The same is true for the rational numbers. If we choose both an upper and a lower bound to a set of rational numbers of any given size, we can always insert as many new ones as we like in between the boundaries. We have a problem here with the notion of completeness, the notion of treating a set with an undefined property as a whole entity. Try picking up a section of hose in the middle when the spigot is off but the nozzle is open and the hose is full of water. What happens? Water may leak out of the nozzle because the hose (set) is not bounded at both ends and the natural numbers (water molecules as a group) have no fully defined shape.

Cantor's demonstration of the uncountability of the real numbers is complicated by several problems with the decimal system. When we consider these problems we discover that his demonstration actually always generates duplicate numbers on the list, just like (n + 1) is just another natural number. The first problem is that all decimals must be expressed in base two or larger. By definition you can not write fractions written in decimal point format in unary base. Using base two as the simplest possible example, we can create an algorithm for the list of decimal real numbers between 0 and 1 so that the 1 digits in the decimals crawl to the right from the decimal point as we go down the list. The first number on the list is 0.00000 . . . . The last number on the list is 0.11111…….., which corresponds to 1.000000……. The decimal places gradually fill to the right with 1's as we go down the list. Eventually we end up with solid 1's going to "infinity" and the decimal odometer flips over and zeroes out.

0.000000……

0.100000…….

0.010000…….

0.110000…….

0.001000…….

……………….

……………….

0.1111111…….

This gives us an infinite list that is not in numerical order, but is quite orderly. However, our base is greater than 1, so the 1-digits in numbers greater than 0.000… crawl slowly to the right as we go down the list. (In the case of higher bases the digits higher than 0 crawl even slower.) Therefore, diagonalizing of this list and flipping of the digits generates a new decimal, but the decimal ends with an infinitely long string of 1's. This brings up the second major problem with decimals. Mathematicians disallow any binary decimal that ends with an infinite string of 1's, because it zeroes out and increments the first 0 that it encounters to the left up to 1. (An analogous situation also holds for any other base.) For example, when we diagonalize in the above list we end up with 0.0011111……. This "number" translates into 0.0100000….., which is the second number on the list. So we start with an infinitely long list that we have designed to be a very thorough catalog of decimals, but when we diagonalize, we end up with a number that seems new at first glance, but turns out to be already on the list when we look more closely. The notion that it is not on the list is an illusion created by ignoring the fact that the decimal system contains endless duplicate numbers that are complementary.

We might imagine that the list we gave is not actually complete. Perhaps it is only a subset of the complete list. Let us consider a list that is randomly ordered and contains endless numbers in all possible combinations that consist of infinitely repeating decimals and infinitely non-repeating decimals with 1's and 0's distributed on average with equal probability at each digit place (e.g., 0.1010101010101…….) Our list contains all possible infinite sequences of 1's and 0's. It therefore must contain all possible pairs of complementary decimals. For example, the complement to 0.1010101…. is 0.0101010….. So both these numbers must be on the list. We do not know the order of the list, but we know it is complete, and we also know that every number on the list will have a complementary number where each 0 becomes a 1 and each 1 becomes a 0. Now let's try to diagonalize. No matter what sequence we find along the diagonal, the flip transformation of that sequence will be – by definition and the rule of Cantor's diagonalizing procedure – the complementary number for that sequence. By the definition of the contents of the list we know that both the decimal produced by Cantor's diagonalizing procedure and the complementary number it transforms into when we flip the values of the digits **must** be on the list and can not be new numbers. For example, if the diagonal decimal happens to be the algebraic non-periodic decimal, 0.101001000100001….., then its flip decimal will be 0.010110111011110…, and both numbers will be on the list **by our definition of the list, which is all we know about the list**. Thus we disprove the validity of Cantor's "demonstration" of the uncountability of irrationals and discover that irrationals are just as countable as natural numbers. We are unable to demonstrate any "infinity" greater than that of the natural numbers when we interpret "infinitude" in the sense of multitude (i.e., "how many"). Does this mean that Cantor's whole theory of transfinite numbers collapses, since it depends on the ability to demonstrate the existence of "infinitude" greater than that of the natural numbers? Perhaps we simply adjust our viewpoint and treat infinity as a dimension. Suppose that the irrational numbers have an additional dimension of infinitude. We can then imagine an entire hierarchy of mathematical sets that possess more and more different properties or dimensions of infinitude. Thus we can count transfinite dimensions just like we count ordinary dimensions. We simply reinterpret our notion of transfinite numbers and perhaps it survives and perhaps takes us in new directions.

The apparent "greater infinitude" of irrational numbers does not mean that there are infinitely many more of them than other infinite sets. It simply means that they are less defined "value-wise" than other types of numbers. They take on a new property of infinitude (non-definiteness). Let us examine this new property.

I call non-algebraic irrational numbers "peanut numbers". This name derives from the Styrofoam peanuts that are used as packing filler. Mathematicians invented irrational numbers to solve the problem of continuity. Several thousand years ago the Greeks already were bothered when they noticed irrational quantities showing up in their beautifully rational system of geometry. For example, they found that the diagonal of a unit square turned out to be the square root of 2. They could not compute an exact value for the ratio of the square's clearly real diagonal line length to the unit side of the square. It was irrational. They also found that they could not compute an exact value for what they called pi, the ratio of a circle's circumference (C) to its diameter (D). In a simple notation we write this ratio as follows: ($\pi = C / D$). The Greeks could represent this symbolically and conceptually, but they could not reduce it to a precise ratio of two whole numbers. The handiest estimate turned out to be 22 / 7. This was close, but no cigar. The appearance of such irrational values in the simplest figures of plane geometry was very frustrating. Mathematicians eventually decided that you can define certain irrational numbers by means of an algebraic expression (e.g., square root of 2) or an infinite series that converges on the specific value of the irrational number as a limit (e.g., $\pi = 4 - 4/3 + 4/5 - 4/7 + 4/9 - \ldots$). And they also were able to show that you can have irrationals that do not have such finitely expressible monikers. And there can be any number of these peanut numbers that can magically appear between any two numbers with recognizable value. What gives here?

The simple answer is that the irrational peanut numbers are **elastic space tokens**. One of the remarkable aspects of mathematics is that we can map numbers to points on a line. This generates a relationship between numbers and space. We use this principle regularly when we take out a ruler to measure the size of an object. A ruler is an arbitrary object on which we make a series of equally spaced marks. Then we arbitrarily assign numbers to the marks and use the marks as a standard gauge for assigning numbers to "measure" the size of some other object. In fact, no object has any particular size. Space is an arbitrary notion that is wholly defined by the observer. People convince themselves that they live in a space of certain dimensions and size by setting up standards of measurement (conventional rulers) and then agreeing to talk about objects that they share from the viewpoint of these standard measurements. Such a notion of space is entirely conventional.

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|.........|.........|.........|.........|.........|.........|
1         2         3         4         5         6
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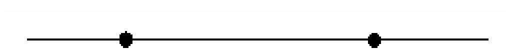
The above is an example of a ruler with 6 vertical marks at equal intervals. In between each vertical mark is a series of 9 dots that mark off smaller intervals, fractions of the larger intervals. The vertical marks represent natural counting numbers, so we can use them as a standard for measuring objects of similar "size". The dots represent rational numbers (fractions) and allow us to measure objects that do not quite match in size the distance between vertical marks. We can insert as many marks as we like between the existing marks, assigning to each an appropriate rational number. We are only limited by the molecular structure of the ruler. At some level of resolution we can no longer mark off finer gradations because we start to fall into the gaps between the physical components of the ruler. Mathematicians pretend that in mental space they can keep making finer and finer gradations, but that also is an illusion. Eventually the whole thing turns to chaotic mush due to the

Uncertainty Principle.

We can also make some marks on our ruler to represent certain algebraic irrational values. For example, we can erect a square over one natural number unit and then rotate the diagonal down until it marks off a point on the ruler that represents the square root of 2. Or we can draw a circle with a diameter of 1 unit and then roll the circle one cycle along the ruler and mark off the value of pi.

Mathematicians define a point as something that has no size. Size is the fundamental property of space. However, if a point has no size, then you can not generate spatial intervals with any number of points – by definition. This leaves the mathematicians in a quandary. If they give a point any size, that becomes "arbitrary" and not "pure" mathematics. If they deny the point any size, then they can not generate the space in which to do geometry with these points. What to do? This is where the irrational numbers come to the rescue.

Look at the ruler again. The marks on the ruler are supposed to represent pure mathematical points that have no size. They are precise and exact. The actual marks on the ruler have size and are not exact. So we have to pretend. Also, the marks are made on a ruler that already exists in something we call space. It has size, and so does every one of its components. The mathematician has scattered size-less points onto a **pre-existent** space and arbitrarily assigned numerical values to them. Now he wants to account for the continuity of the space. He can not do this with his points alone. He needs something else. That something else is peanut numbers. Another name for them is "gap" numbers. Another name is irrational numbers.



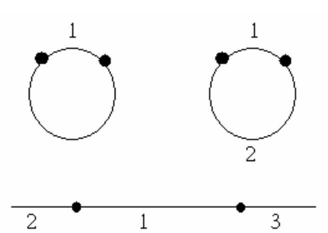
Euclid decided that two points determine a "straight" line. Above you can see two large points that represent the definition of a line. Through the two points I have drawn the line. The two points can represent the end points of a line segment or interval on the line. Or they can define the position of the line in undefined space. We can add more points along the line – as many as we like – but two is the minimum requirement. The points we add may or may not keep our line "straight". The notion of straightness remains a bit fuzzy and starts to depend on other fuzzy notions such as extension. The problem of co-linearity is an interesting issue we will not discuss here.

The key thing to notice here is that the two points we choose must not be coincidental.

They can not be in the same place in space. Are two pure mathematical points that exist in the same place still two points, or do they become indistinguishable from one point? What do you believe?

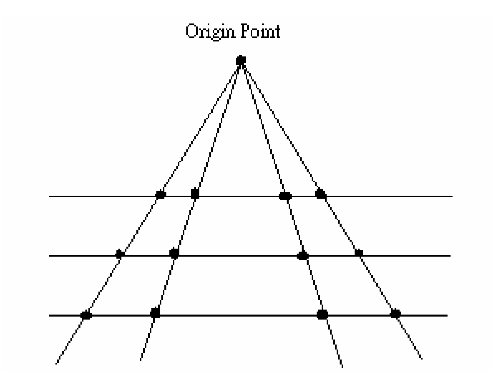
Here we come to an important discovery. Euclid can not simply say that two points determine a line. The two points must be non-coincidental. It turns out that by simply making the two points non-coincidental we assume a space between them. And the assumption of that interval of space generates the entire set of real numbers and the points that correspond to them. So we have 1, 2, . . . infinity. The infinity is an undefined space that exists between the two points and beyond them in all possible directions. This is the first peanut number. We can assign a number to each point, but we can not assign an ordinary number to the undefined space, because that space is not a point without size. It is something else that has size, and that size is defined by the two points that bound it into an interval or gap. It is not a rational (or algebraic/geometric) number, because we have defined the points on a line as the loci of rational numbers. So we label the spatial interval as an irrational number.

The funny thing is that no matter how many points we mark off on our ruler, we will always find gaps between the points. These gaps are the irrational numbers. You can see now why irrational numbers are quite countable, just like regular numbers. For example, there is only one gap number **between** the two points on our primordial line. We can take the gap between the two points to include all possible space in all directions outside the points as well as the space between the two points. Or we can think in the mode of projective geometry and consider there to be 2 gaps between the 2 points: the finite "inside" interval that marks the **shortest** "straight-line" distance between the two points, and the infinite interval that wraps around the "outside" portion of the line to mark the **longest** "straight-line" distance between two points. Or we can think of the Euclidean line as having a gap "beyond" the first point, a gap "between" the two points, and a gap "beyond" the second point. In each of these cases the gap numbers are quite countable because our points are also quite countable. I think the projective viewpoint is the most reasonable, since it recognizes that an inside must have an outside in order to exist as an inside and gives the same number of gaps as there are points.



If we have an indefinite number of points, then we have an indefinite number of gaps. But we know that however many points we have, there will be either the same number of gaps, or one more, or one less, depending on the viewpoint we take. The gap numbers have an additional dimension of indefiniteness. They have an indefinite size. Point numbers always have the same size – no size at all. Gap numbers automatically resolve the problem of continuity, because they automatically fill the gaps between point numbers. The indefinite size of a gap number gives us the freedom to define that parameter however we like. This means that space is a notion that is up to the observer to define any way he likes. In some forms of math the irrational gap number is represented by the undefined notion of a **neighborhood**. Obviously we can label every neighborhood with the numerical value of the point that defines it. The spatial property of size distorts the repeating decimal into a random, non-repeating decimal. Randomness is chaos. Since by definition there is **no way** to represent infinite chaos in a finite manner, we **believe** it into existence by adding the spatial indefiniteness of the neighborhood concept and treat it as an irrational number. It really is just the gap between two points.

A remarkable result of this property of gap numbers is that space necessarily becomes completely elastic. Mathematicians marvel that a line interval of any size as if by magic contains the same number of points.



The above diagram shows three parallel lines cut by diagonal lines that radiate from a common origin point. We can draw as many lines as we like from the origin point so that they intersect the parallel lines. Each line from the origin point will intersect all three parallel lines at a different point on each parallel line. The trick that makes this possible is that each of the parallel lines has its own set of gaps between the intersecting points. There is a one-to-one correspondence between the gaps along one parallel line and any other parallel line in the figure. The only difference is in the size of the gaps. What happens here is a shift in scale. Points can not shift their scale because they have no size. Only gaps can shift in scale because they have the property of size.

Once we understand this principle of gaps with no-limit spatial elasticity, the mysterious "uncountability" of irrational numbers goes away. So does the problem of continuity. The gaps perfectly link points together. There are no gaps in the gaps because the gaps already are gaps. Only points can function as gaps between gaps. Thus points and gaps are complementary types of "number".

The basic principle in this whole system is that of freedom. Infinity is the property-less "property" of lacking definition. We find it in its ultimate condition only by dropping **all** definitions. Infinity is the freedom to set our own definitions on whatever creations we choose to create. "Infinity" therefore is the essential property of the Self. The Self in its essential nature is totally undefined. It is totally unknown. Ironically most people are afraid of the unknown, not realizing that they are simply afraid of who they really are – infinite potential. This is the state of freedom. The Self can then choose to define itself any way it prefers. Of course, this may include the creation of multiple viewpoints and the creation of conventions so that the similarities and differences of viewpoints can be shared. A viewpoint is a point from which an observer can view. It generates the possibility of perspective.

The peanut gap numbers mathematically represent our freedom to define our space any way we want to. We deliberately keep them undefined so we can use them with total freedom. Gap numbers do not even necessarily have to generate space. For example, they can generate time. Simply choose two moments in time that are not in the same instant and you generate an interval of time. This is a gap. The gap may be in any dimension you please. Even the properties of a set of gaps are totally elastic. The points tell you where you are, and the gaps tell you wherein you are.

In addition to Cantor's famous "demonstration", there is another method mathematicians sometimes use to demonstrate the supposed non-denumerability of irrationals. First we suppose that we have put all the points on a line segment between point 0 (the starting point) and point 1 (the ending point) in a sequence:

- $a_1, a_2, a_3, \ldots\ldots$

We create an arbitrary interval whose length we set at 1/10, and place the first number on our list at some point in that interval. Then we create an interval of length 1/10^2 (i.e. 1/100) and place the second number on the list inside it at some point. Then we make an interval of 1/10^3 (i.e. 1/1000) and place the third number on the list inside that interval. We continue in this manner until we cover all the points on the list. The sum of all the intervals happens to be the geometric series that converges on the value:

- $1/10 + 1/100 + 1/1000 + \ldots = (1/10)\,[1 / (1 - [1 / 10])] = 1/9.$

So the sequence of all real numbers that form the interval of 1 unit in length can be put in an interval of 1/9. Mathematicians find this "intuitively absurd" (e.g., Courant and Robbins, p. 83.) The particular value of the intervals we chose for the demo is arbitrary. We could have used 1/2, 1/2^2, 1/2^3, . . . as our set of intervals or any such series that converges. The choice of 1/10 means we are using base 10. By shrinking the initial interval we can shift our parallel lines closer and closer to the "origin point". When the diagonals all converge at the origin point we discover that the set of points has a measure of zero. The interval is an illusion created by the observer.

From our discussion above you can easily see that there is nothing absurd about this interval demonstration. It fully agrees with the diagram we made above that demonstrates the elasticity of space. We can scale the gaps any way we want and a complete interval is still a complete interval. In this demonstration our tricky mathematician has simply reversed the roles of the point and the neighborhood to confuse the unwary. What he calls the "interval" is really the digit place in a decimal. The sum of his intervals actually represents a single infinite decimal number. Each digit place represents a single point without size. The sum does too. Of course, since the points have no size, any sum of them has no size, so you get a meaningless answer regardless of the gaps. The infinitely overlapping points show us a model of the "origin point" on our previous drawing of scaling intervals. The "number" that our mathematician places as a point within each "interval" actually represents the "real" interval, the gap between the points that define the digits. We do not need to know the exact value of the gap number (its non-periodic decimal sequence, which in any case is unknowable by definition). We only need to know for sure that each one is different. Its being in a separate interval assures us of this. Once we insert the gap neighborhoods between the points in our sum of points, we can scale the origin point into a line of any size we like. With another transformation the line becomes a plane or a 3-space, and so on. Thus we find that this little demonstration shows us once again how we can generate an n-dimensional space from a single point.