Chapter 1. Riddle of the Sphinx: The Mysterious Role of Mathematics in Physics.

In the ancient world there was a myth about a creature with a human head and a lion's body called the Sphinx. In Greek tradition this creature was a strange hybrid between an angelic woman (sometimes even endowed with wings) and a powerful lion. If you want a feel for this creature, take a look at the traditional Tarot trump card for Strength. It shows a calm and beautiful lady taming a lion with her bare hands. You will also sometimes find the Sphinx depicted on top of the Wheel of Fortune card in many Tarot decks. Or you may go to Egypt and stand by the great pyramids. It is said that the Sphinx would pose riddles to people, and, if they could not answer the riddles, the Sphinx would then devour them.

Exercise: Do you remember the Sphinx's riddle that Oedipus solved? What might it have to do with Observer Physics?

It occurs to me that the Sphinx herself is an appropriate symbol for the mysterious question of how mathematics comes to be such a powerful tool in the study of physics. Mathematics is a beautiful angel of the Mind, and physics is a powerful beast that commands a World of dynamic motion and force. Beauty and the Beast. Beauty calmly and effortlessly tames the Beast.

How is it that the elegant beauty of a purely mental discipline can handle the wild physical world with such precision, confidence, and accuracy?

In short, why is mathematics so useful in the study and application of physics?

Mathematics is an abstract language for describing mental constructions. Why should it be useful in constructing models that describe the physical world and even predict its behavior? Is there a subtle link between the structure of the mind and the structure of the physical world?

Let us begin with an understanding of what mathematics is.

Mathematics is a language plus reasoning; it is like a language plus logic. Mathematics is a tool for reasoning.' Richard Feynman, 1918-1988 (Physicist), "The Character of Physical Law", Lecture Series, 1964.

First of all mathematics is a language. What is a language? A language is a system for communicating ideas. In its bare bones a language consists of a vocabulary of basic elements, and a grammar or set of rules for organizing the vocabulary in different ways. The set of rules is really a collection of relations and operations. Relations are ways of linking vocabulary elements into expressions, and operations are ways of transforming one expression into another expression. So mathematics, at its basis, is not really about numbers. It is a communication system for describing various relationships and transformations of those relationships.

In a landmark article the linguist Charles Hockett ("The Origin of Speech", **Scientific American**, 203/3 (1960), 89-96) identified a number of design features that are involved in the construction of languages. Primitive communication systems (for example, various animal systems) have only a few design features. More sophisticated systems add more design features. This is true for mathematics as well.

Let's outline the basic design features that are found in human language. I classify Hockett's design features into three categories of three features each. My definitions of the features follow Hockett closely, with perhaps some difference in how I view duality and arbitrariness.

Since human language requires a minimum of two interacting participants who agree on the meaning of a communication, it is ruled by what in mathematics is called a dyadic equivalence relation. Dyadic means a relationship between two participants. Equivalence means that the communication between the two parties is equal -- theoretically they both can agree and understand each other equally well, and share the same information. Dyadic relations such as equivalence have three properties: reflexivity (a = a), symmetry (a = b; b = a), and transitivity (a = b; b = c; a = c). In the following outline we show how the design features of any full-fledged human language, including any richly developed mathematical system, spring from these properties. This is by no means the end point of language development or of mathematical development, but it is a beginning that provides a rich enough framework for lots of exploring.

1. **The Language Community**
   a. **Interchangeability:** A symmetric language community of individuals exchanging information.
   b. **Total Feedback**: A reflexive system that allows a communicator to monitor his message, editing and refining it through techniques such as revision, iteration, and redundancy.
   c. **Traditional Transmission**: A transitive system for accumulating, preserving, and transmitting expressions from generation to generation. Each individual is only born with language ability and must learn to use a specific grammar(s) and lexicon(s).

2. **The Content (Semantics)**
   a. **Displacement**: transitive displacements in various dimensions such as time, space, reasoning, extrapolation, and truth.
   b. **Semanticity**: a reflexive meaning inherent in communications. An expression means what it means.
   c. **Duality**: a lexicon that symmetrically maps signs to corresponding significants and vice versa by means of definitions. Definition can make an abstract element concrete. Different semantic systems (interpretations) may result in different dictionaries. Definitions are fundamentally circular.

3. **The Message System**
   a. **Arbitrariness**: a symmetric modality for shifting communication from medium

to medium. (For expressing a message, one medium is as valid as any other, and the message may be translated back and forth between media. Hence the means of communication is arbitrary. This feature ranges from choosing among synonyms to choice of medium.)

b. **Discreteness**: a vocabulary of primitive (often undefined), self-reflexive, and discrete elements for constructing expressions. The elements can be distinguished as separate from each other.

c. **Productivity**: Syntax provides a productive set of rules for transitively generating a rich universe of expressions. These rules are operational procedures for establishing and transforming relations between discrete elements.

My use of the = sign to represent the dyadic relation is not exactly the same as a mathematical equivalence relationship. You can translate a sentence in language ***A*** into language ***B***, and then from ***B*** into language ***C***. The meaning will be the same as the sentence in language ***A***. That is transitivity. So the features I call symmetrical (interchangeability, duality, and arbitrariness) also seem transitive. But the key point with them is that you can go from ***A*** to ***B*** and from ***B*** back to ***A***. The key point with Tradition, Displacement, and Productivity is that you can keep on going, transmitting from one to another. So when senders keep passing a message on, that is Tradition, not Interchangeability. When you keep translating to different languages, that becomes productivity, not arbitrariness. Any use of language tends to involve several or all of the features used all together at once, so the features overlap or bunch even though they are distinctly different features.

There is a tenth feature that seems to be a universal human feature. That is **Specialization**, the ability to perform other operations while carrying on communication. When a bee dances, his whole being is involved in communication. He can't do anything else. But we can hold a conversation while driving a car. This is a very useful talent.

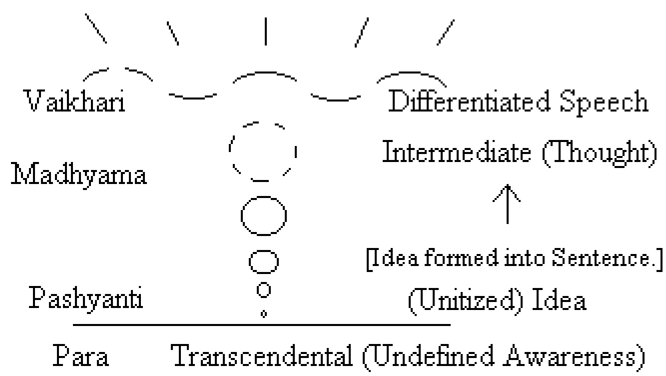
Hockett mentions other features such as broadcasting, vocal-auditory channel, and rapid fading, but these features are characteristic only of special types of human communication and are not generally found all the time. They are not universal design features in human language, but are "localized". For example, speech fades rapidly, but not carvings on stone. Spamming involves broadcasting, but a private conversation (usually) does not. The vocal-auditory, broadcasting, and rapid fading features are actually sub-features of the choice of medium. Arbitrariness ranges from the choice of items to map in the lexicon to the medium that is chosen. Thus arbitrariness is closely related to duality, though still a separate feature.

It is fun and illuminating to examine various communication systems (including many animal systems) and identify what features they include. Mathematics, of course, qualifies as a full-fledged human language system. Numericity (the concept of numbers and counting) is not a separate feature in my opinion. It is a composite expression of several features. For example, the generation of numbers in set theory is an exercise in productivity. The notion of set itself is an idea (semanticity). So is cardinality.

Selection of symbols for numbers is duality. Numbers are written in an arbitrary medium and manner. The primitive elements of a postulational mathematical system exhibit discreteness even though they may contain notions of continuity.

From the above analysis, it is clear that the observer plays a fundamental role in the language of Mathematics. The observer not only creates all the components and expressions, he also determines all of the features he chooses to use and how he uses them. The design features may occur in clusters or all at once.

According to ancient Indian linguistics the diverse features of language emerge from a unitized thought bubble called "sphota". Sphota emerges from "para", or undefined awareness.



Bhartrihari's Theory of the Formation
of Undefined Awareness into a Thought
Bubble (Sphota) that bursts into Speech.

A unique aspect of mathematics, similar to music, is that in its pure form it often has no specific interpretation or "practical world" semantic meaning. We say it is abstract. Pure mathematical objects are mental creations that are only weakly defined or wholly undefined in the physical world. This generalization power is deliberate and may be a combination of semanticity and displacement. Interpretations of abstract mathematical systems lead to models and applications, but such extensions are optional. The "meaning" of pure mathematics can be apprehended in an abstract, aesthetic way simply as patterns of relationships and their orderly transformations.

The "practical" usefulness of any mathematical system in physics comes from the creation of an interpretation that applies the mathematical system as a model for physical world phenomena. This is the process of mapping a message system to a physical situation. Physics further has a theoretical and applied aspect, each with its own mathematical regimen. Given these definitions, how can mathematics serve physics (and other sciences) so precisely and powerfully?

Mathematics is precise thinking. A mathematical system is a precisely defined and as much as possible logically consistent language for expressing sets of abstract beliefs. Beliefs can be projected into experiences. Experiences are nothing other than strongly held beliefs. (You may say that the previous sentence expresses my strongly held belief.) "Therefore" it is natural that we should be able to articulate in thought and language our beliefs that are expressed in the form of physical systems. The only discrepancy between a mathematical description and the corresponding physical system might arise due to "transparent" beliefs that have been overlooked or wrongly selected beliefs that are not consistent with the physical system we are experiencing and describing.

A number, algebraic expression, or relationship can be very exact. Ordinary language is usually relatively imprecise thought. The previous sentence is an example, I suppose. So is that one. Let me show you how levels of CERTAINTY -- which correspond to levels of precision -- are reversed by the mirror (or perhaps lens) of observation that resides in the gap between observer and observed. The discrepancy with regard to certainty arises from an interaction between the mental principle of numericity and the physical quantum principle.

In the mental world of numbers, integers and rationals are discrete, carrying precisely certain values (infinitely precise?) and sequence. Non-algebraic irrational numbers, on the other hand, are not precisely definable. We can not be certain what the nth digit of such a number is. For example, we know that the nth digit to the right of the decimal dot after the whole number 5 is 0, and the nth decimal digit of 1/3 is 3. We can't do this for hardcore irrationals. They are fundamentally uncertain with regard to their values.

Irrational numbers can be viewed as infinite mental wave forms. Their component digits get "smaller" by orders of magnitude as they go off into the mental "distance". They can be written down symbolically in part. This part may be complete if the sequence of digits "terminates" and then degenerates into an infinite string of 0's.

In the world of quantum physics, which we currently assume is the basis for our world of objective experience, we find that the continuous wave function of a quantum particle is precise and certain, but the discrete quantum particles that we observe are fundamentally uncertain in terms of position and momentum -- and this is borne out by numerous experiments. Such a situation puts physics in a quandary when it uses mathematics to describe physical systems. We habitually use integers to count photons, electrons, and other such particles. But their locations and/or momenta (i.e. how we define them in terms of mass, velocity, position, etc.) are inherently fuzzy because of Planck's constant and difficulties with measurement. Mathematically we can only predict outcomes as statistical probabilities rather than as certainties even though the wave function may be smooth and continuous.

Wave functions tend to be continuous probability distributions spread out over time and space. Probability is often expressed by a value between 0 and 1. The wave function tells us the probability for a certain particle or event to occur at any point in space or time. Most real functions are continuous. For example, here is a definition of a function from a standard calculus text (Edwards and Penney, 2nd ed., p. 9.)

* A real-valued function f defined on a set D of real numbers is a rule that assigns to

each number x in D exactly one real number $f(x)$.

Functions may be discontinuous and not necessarily real-valued. Quantum wave functions are generally expressed in the complex domain. This is a set that contains the reals as a subset. In any case any differentiable function must be continuous, although there are continuous functions (such as certain fractals) that are not differentiable.

Real-valued and complex valued functions are mostly comprised of irrational values, yet the evolution of the qwiff (quantum wave function) is fully determined and certain once the initial conditions are set. So we use certain discrete numbers to construct mental images of uncertain physical objects and fuzzily uncertain numbers to construct our mental images of physical systems as functions that describe a system in full with a certainty of 1 (100%). Most of us, including physicists, live our lives all twisted and misled by appearances.

People tend to look at the world upside down through the funhouse mirror of the mind, where the simple counting numbers are supremely reliable. That is why people mistakenly put so much trust in physical objects. Integers are trustworthy beyond all time and space, and they also resemble abstract objects. So people usually "calculate" the objects in their world using trustworthy natural whole numbers, integers, and rationals.

We say, "I saw 4 chairs."
We do not say, "I saw 4.29683651287532746389236478301920037 84... chairs."

We count one apple, two apples, three apples. Tomorrow or next year the numbers 1, 2, and 3 will still be around, but those apples will be gone. People should really trust the qwiffs that they can't see, and not the apples. Of course, it's OK to talk about 2 apples if you are going to eat them today and then move on. So this rough mapping works well enough in a limited reference frame. But it breaks down as soon as you stretch the boundaries of the reference frame.

The qwiffs tell the evolution of the apple wave function over time. The particles that make the apples are only symptoms of the presence of qwiffs. When you count apples and oranges, you have an illusion of a reliable count because of the quantum statistics of so many atoms. But it is an illusion nonetheless. If you expand your frame of time and/or space that you observe them in, they get fuzzy very fast.

Look at a banana under an electron microscope. Leave it sitting in the sun for a few weeks. Where is it? The discoveries of quantum mechanics invite physicists to de-invert the mirror that they use to observe the world. Perhaps the qwiffs should be dealt with as rationals, algebraic numbers, or even whole numbers and the quantum particles counted as non-algebraic irrational "peanut" numbers. A farmer with an apple orchard understands the quantum wave function for apples and organizes his reality so that he has an abundance of apples for himself and his community even though he is usually unable to predict who will end up eating which apple from his orchard.

Here is how **continuity** is defined in the same text, p. 63.

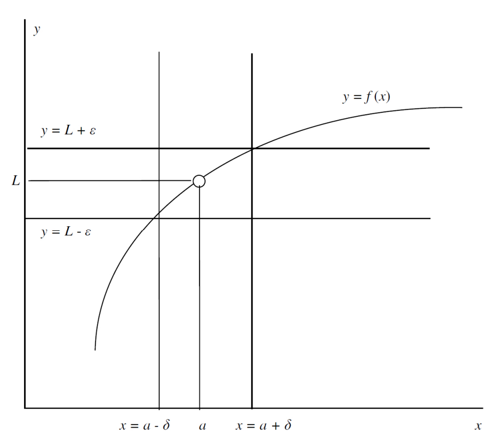
* Suppose that the function *f* is defined in a neighborhood of *a*. We say that *f* is **continuous at *a*** provided that lim ($x \to a$) $f(x)$ exists and, moreover, that the value of this limit is $f(a)$.

This notation tells us that *x* can assume any value as close to *a* as we like, so long as *x* does not equal *a*. The limit is that $x = a$, but that value is not included in the range of the lim ($x \to a$) $f(x)$. However, the function $f(x)$ also has a value $f(a)$ at *a,* because it is defined in a neighborhood of *a* even though the point *a* is excluded from the limit. The continuity is derived from the ability to "approach" the limit point as close as you like with all values that differ by even infinitesimal values, most of which are irrational, from the limit point $f(a)$ that corresponds to the point $x = a$. This is such a contorted way of defining continuity, and is much more complicated, confusing, and even misleading, than is necessary.

I call irrationals "peanut" numbers because they are like the styrofoam peanuts used as packing fill. Mathematicians made them up to pack intervals until they became continuous. No mathematician can write down a single "peanut" number or build continuity from the dimensionless discrete "point" components he starts with. Look again at the above definition of continuity. The mathematician must use the concept of limits to convey the idea of continuity. This involves an infinite sequence of values ($x \to a$). In the real world there are no infinite sequences.

Somewhere, at a superfine interval, the mathematician makes a "quantum leap". He gives up going to smaller and smaller neighborhoods of *a*, and jumps over to the limit. Here is a traditional definition of limit using *epsilons* and *deltas*, as given by Edwards and Penney, p. 41.

* "We say that the number *L* is the **limit** of $F(x)$ as *x* approaches *a* provided that, given any number $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|F(x) - L| < \varepsilon$ for all *x* such that $0 < |x - a| < \delta$."



So the concept of a limit is that you can sneak up on it by taking values of x that get as close as you like to a, **but not equal to *a***. Then the corresponding value of $F(x)$ will get correspondingly close to L. To get to a you have to jump beyond the series of "approaching" values of $0 < |x - a| < \delta$. The problem here is that when $x = a$, you get 0, so you must not let it get there, because of the requirement $\delta > 0$. So we have to "jump" to get to the **limit** of the process.

The strange situation of pinning down a "limit" is further complicated by the use of the word "neighborhood" in the above definition of continuity. I could not find a definition for "neighborhood" in Edwards and Penney, although they do speak of a "deleted neighborhood". So we define the foundation for calculus using a key term that lacks a definition. This is a general property of mathematical systems -- they are based on a small number of discrete primitive elements that are undefined. When you try to look squarely at them, they "fuzz out" completely on you. Furthermore, the notion of a deleted neighborhood (we don't allow x to become exactly equal to a) is used to justify an expression that otherwise results in division by 0 at the limit when $x = a$, a situation that causes the limit to explode into -- you guessed it -- INFINITE UNCERTAINTY.

The tricky part is that the mathematicians like numbers to have no physical size. They are infinitesimally small "point-size" objects. The complete set of real numbers written as infinite decimals can be represented in a geometry interpretation by the points on a line segment between 0 and 1. (This segment can stand in for the entire Real Line. Real number sets have a holographic property that they reflect the whole in the part, something that is not true in the physical world.)

* $F(x) = f(x) - f(a) / x - a.$

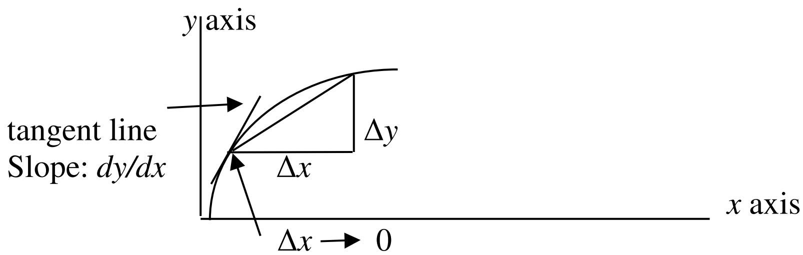
This is the basis for calculus -- infinite uncertainty. And yet it results in extremely precise and marvelous calculations. Very odd.

"More intuitively, we can say that if we want to get all the $f(x)$ values to stay in some small neighborhood around $f(c)$, we simply need to choose a small enough neighborhood for the x values around c, and we can do that no matter how small the $f(x)$ neighborhood is; f is then continuous at c."
(**Wikipedia**, "Continuous function")

Historical Note (**Wikipedia**, "Continuous function"):
A form of the epsilon-delta definition of continuity was first given by Bernard Bolzano in 1817. Augustin-Louis Cauchy defined continuity of $\{y=f(x)\}$ as follows: an infinitely small increment a of the independent variable x always produces an infinitely small change $\{f(x+a)-f(x)\}$ of the dependent variable y (see e.g., *Cours d'Analyse*, p. 34). Cauchy defined infinitely small quantities in terms of variable quantities, and his definition of continuity closely parallels the infinitesimal definition used today (see microcontinuity). The formal definition and the distinction between pointwise continuity and uniform continuity were first given by Bolzano in the 1830s but the

work wasn't published until the 1930s. Like Bolzano,[1] Karl Weierstrass [2] denied continuity of a function at a point *c* unless it was defined at and on both sides of *c*, but Édouard Goursat [3] allowed the function to be defined only at and on one side of *c*, and Camille Jordan [4] allowed it even if the function was defined only at *c*. All three of those nonequivalent definitions of pointwise continuity are still in use.[5] Eduard Heine provided the first published definition of uniform continuity in 1872, but based these ideas on lectures given by Peter Gustav Lejeune Dirichlet in 1854.[6]

The invention of zero was a great advance in mathematics. It expanded and facilitated the performance of arithmetical calculations. But there was a problem when you tried to divide by zero. To patch things up, division by zero was outlawed. Newton showed that this outlawing of division by zero was an overreaction. He broadened his viewpoint and generalized the operation of division by zero. In so doing he discovered that the problem of division by zero only really occurs in special cases. The triumph of this insight was his discovery of the calculus and a beautiful way of resolving the paradox of Zeno.



We shall draw a curved figure and organize that figure in space by imposing an *x-y* coordinate Cartesian grid over it. We can now write an equation in terms of (*x*) and (*y*) to describe the figure. The operation of division can be represented as a ratio of the rise (Δ*y*, the distance between two chosen points on the curve as measured along the *y* axis) to the run (Δ*x*, the distance between the same two points as measured along the *x* axis) for any two points in our curved figure in terms of values on our *x-y* grid. This ratio of two distances is called the "slope", since that is just what it looks like -- the slope of a terrain. It tells relative to the grid whether you are headed upward or downward and by how much "vertical" rise per "horizontal" run. With the rise and the run the segment of curve usually forms a roughly triangular shape. Newton studied the behavior of these "triangle-like" slopes and discovered that they more or less resemble right triangles. In general, the bigger the slice of the curvy drawing we consider, the less the curved segment resembles the corresponding triangle's hypotenuse (diagonal long side that cuts through the curve at the two points). We can bring one of the two points on the curve closer and closer to the other point. Then the slices of the curve made at the two points define shorter and shorter segments of the curve and those segments get closer and closer to a corresponding hypotenuse on a perfect triangle. But the triangle also gets smaller and smaller as the two points on the curve converge. The idea dawned that there could be a **limit** at which the two points on the curve become essentially identical. That limit is reached when the run (Δx) is reduced to zero, and often is written with the symbol *dy/dx*. Since the slope ratio is the rise over the run, this is like dividing by zero. Although the textbooks try to tell you this is **not really** division by zero, it sure looks like it. (For example, see Edwards and Penney, p. 33: "It is also important to understand that *dy/dx* is a single symbol representing the derivative and is not the quotient of two separate quantities *dy* and *dx*.")

To his surprise Newton found out that reducing the "run" to zero did not cause the ratio to explode! It only "explodes" into an indeterminate value when you think of the denominator of the ratio in terms of the value 0 instead of as an **interval**. By just slightly shifting his viewpoint, Newton found that there is a whole class of possibilities that occurs in the limit of a ratio of two infinitesimally small numbers, and only in one case does the ratio seem to explode, and that case makes sense too.

This was the value of **visualizing** a model. He could see that something was really there. If you inscribe a right triangle inside a curve and then squeeze its (Δx) interval closer and closer together along the curve, the diagonal hypotenuse that represents the slope of the triangle squeezes up closer and closer to the curve. When (Δx) reaches zero, the slope of the "triangle" becomes the tangent to the curve where the two points of (Δx) converge into a single point. From this Newton discovered the principle of the derivative.

When Δx is greater than 0, the slope is a line that cuts the curve at two points. When Δx becomes 0, the slope of $\Delta y/\Delta x$ becomes *dy/dx* but still has value and becomes a single point on the curve in the line with slope tangent to the curve at that point. However, **the slope is measured NOT from the curve, but from the tangent line.** The tangent is a straight line and has a constant slope, so any two points along it give the proper answer.

Here is what Newton found: $dy/dx = nx^{n-1}$.
* $y = x^3$; $dy / dx = 3\ x^2$ = a variable nonlinear slope.
* $y = x^2$; $dy / dx = 2\ x$ = a variable linear slope.
* $y = x^1$; $dy / dx = 1$ = a constant slope.
* $y = x^0$; $dy / dx = 0$ = a flat line with zero slope.
* $y = n$; dy / dx = undefined = infinite slope, no slope at all -- a vertical line.

These mathematical expressions all have graphical representations. Not only did nothing explode (except, understandably, in the last case), Newton had found a way of determining an "instantaneous" or point value ratio in a varying situation. He also demonstrated the obvious fact that Achilles does beat the tortoise. The rest is history, and lots of textbooks full of formulas.

The secret to Newton's discovery is that we are not really considering an interval of zero displacement on a curve. Newton has **shifted attention** from two points on the curve defined by his function and he has jumped to the slope of a line that happens to be tangent to the curve, touching it only at one point on the curve. This is a "mental quantum leap" that Newton and subsequent calculus promoters make with an unannounced shift of definition and viewpoint with regard to the model they are proposing. So the whole business of sliding points on the curve is totally irrelevant to the problem at hand**.** This problem does not involve any sliding points, infinitesimal values, neighborhoods, and so on. **The task is simply to choose a point on the curve and then calculate the slope of the straight line tangent that touches the curve at that point.**

The values embodied by *dy/dx* are from the tangent line, not the curve and can be determined by any two displacements along the x and y axes in terms of the **tangent line**. So we do not need to fuss around with squeezing distances on the curve. We can pick any arbitrary point on the curve and then find a way to calculate the value of the tangent slope at that point on the curve.

For example, if the curve's function is $y = x^2$, we simply need to find a differential (slope) that stays constant for that tangent line and passes through the given point on the curve. **The slope of a straight line is by definition constant**.

Miles Mathis has written an article "Calculus Simplified" in which he demonstrates that it is not necessary, or even advisable, to have a theory of limits as the foundation for the differential calculus. All you need is some tables of the powers of various integers, something ancient cultures were already well aware of. In this excerpt I quote from the article I give the basic tables compiled by Mathis with some of his explanatory comments.

1	Δz	1, 2, 3, 4, 5, 6, 7, 8, 9….
2	$\Delta 2z$	2, 4, 6, 8, 10, 12, 14, 16, 18….
3	Δz^2	1, 4, 9, 16, 25, 36, 49 64, 81
4	Δz^3	1, 8, 27, 64, 125, 216, 343
5	Δz^4	1, 16, 81, 256, 625, 1296
6	Δz^5	1, 32, 243, 1024, 3125, 7776, 16807
7	$\Delta\Delta z$	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
8	$\Delta\Delta 2z$	2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2
9	$\Delta\Delta z^2$	1, 3, 5, 7, 9, 11, 13, 15, 17, 19
10	$\Delta\Delta z^3$	1, 7, 19, 37, 61, 91, 127
11	$\Delta\Delta z^4$	1, 15, 65, 175, 369, 671
12	$\Delta\Delta z^5$	1, 31, 211, 781, 2101, 4651, 9031
13	$\Delta\Delta\Delta z$	0, 0, 0, 0, 0, 0, 0
14	$\Delta\Delta\Delta z^2$	2, 2, 2, 2, 2, 2, 2, 2, 2, 2
15	$\Delta\Delta\Delta z^3$	6, 12, 18, 24, 30, 36, 42
16	$\Delta\Delta\Delta z^4$	14, 50, 110, 194, 302
17	$\Delta\Delta\Delta z^5$	30, 180, 570, 1320, 2550, 4380
18	$\Delta\Delta\Delta\Delta z^3$	6, 6, 6, 6, 6, 6, 6, 6
19	$\Delta\Delta\Delta\Delta z^4$	36, 60, 84, 108
20	$\Delta\Delta\Delta\Delta z^5$	150, 390, 750, 1230, 1830
21	$\Delta\Delta\Delta\Delta\Delta z^4$	24, 24, 24, 24
22	$\Delta\Delta\Delta\Delta\Delta z^5$	240, 360, 480, 600
23	$\Delta\Delta\Delta\Delta\Delta\Delta z^5$	120, 120, 120

From this, one can predict that

24 $\Delta\Delta\Delta\Delta\Delta\Delta z^6$ 720, 720, 720

And so on.

This is a table of differentials. The first line is a list of the potential integer lengths of an object, and a length is a differential. It is also a list of the integers. . . . After that it is easy to follow my method. It is easy until you get to line 24, where I say, "One can predict that. . . ." Do you see how I came to that conclusion? I did it by pulling out the lines where the differential became constant.

7	$\Delta\Delta z$	1, 1, 1, 1, 1, 1, 1
14	$\Delta\Delta\Delta z^2$	2, 2, 2, 2, 2, 2, 2
18	$\Delta\Delta\Delta\Delta z^3$	6, 6, 6, 6, 6, 6, 6
21	$\Delta\Delta\Delta\Delta\Delta z^4$	24, 24, 24, 24
23	$\Delta\Delta\Delta\Delta\Delta\Delta z^5$	120, 120, 120
24	$\Delta\Delta\Delta\Delta\Delta\Delta\Delta z^6$	720, 720, 720

Do you see it now?

$2\Delta\Delta z = \Delta\Delta\Delta z^2$

$3\Delta\Delta\Delta z^2 = \Delta\Delta\Delta\Delta z^3$

$4\Delta\Delta\Delta\Delta z^3 = \Delta\Delta\Delta\Delta\Delta z^4$

$5\Delta\Delta\Delta\Delta\Delta z^4 = \Delta\Delta\Delta\Delta\Delta\Delta z^5$

$6\Delta\Delta\Delta\Delta\Delta\Delta z^5 = \Delta\Delta\Delta\Delta\Delta\Delta\Delta z^6$

All these equations are equivalent to the magic equation, $y' = nx^{n-1}$. In any of those equations, all we have to do is let x equal the right side and y' equal the left side. No matter what exponents we use, the equation will always resolve into our magic equation.

I assure you that compared to the derivation you will learn in school, my table is a miracle of simplicity and transparency. Not only that, but I will continue to simplify and explain. Since in those last equations we have z on both sides, we can cancel a lot of those deltas and get down to this:

$2z = \Delta z^2$

$3z^2 = \Delta z^3$

$4z^3 = \Delta z^4$

$5z^4 = \Delta z^5$

$6z^5 = \Delta z^6$

Now, if we reverse it, we can read that first equation as, "the rate of change of z squared is two times z." That is information that we just got from a table, and that table just listed numbers. Simple differentials. One number subtracted from the next. Given an x, we seek a y', where y' is the rate of change of x. And that is what we just found. Currently, calculus calls y' the derivative, but that is just fancy terminology that does not really mean anything. It just confuses people for no reason. The fact is, y' is a rate of change, and it is better to remember that at all times.

What does it *mean*? Why are we selecting the lines in the table where the numbers are constant? We are going to those lines, because in those lines we have flattened out the curve. If the numbers are all the same, then we are dealing with a straight line. A constant differential describes a straight line instead of a curve. **We have dug down to that level of change that is constant, beneath all our other changes.** As you can see, in the equations with a lot of deltas, we have a change of a change of a change. . . . We just keep going down to sub-changes until we find one that is constant. That one will be the tangent to the curve. If we want to find the rate of change of the exponent 6, for instance, we only have to dig down 7 sub-changes. We don't have to approach zero at all.

We have flattened out the curve. But we did not use a magnifying glass to do it. We did not go to a point, or get smaller and smaller. We went to sub-changes, which are a bit smaller, but they aren't anywhere near zero. In fact, to get to zero, you would have to have an infinite number of deltas, or sub-changes. And this means that your exponent would have to be infinity itself. Calculus never deals with infinite exponents, so there is never any conceivable reason to go to zero. We don't need to concern ourselves with points at all. Nor do we need to talk of infinitesimals or limits. *We don't have an infinite series, and we don't have any vanishing terms.* We have a definite and limited series, one that is 7 terms long with the exponent 6 and only 3 terms long with the exponent 2.

The magic equation is just an equation that applies to all similar situations, whereas the specific equations only apply to specific situations (as when the exponent is 2 or 3, for example). By using the further variable "*n*", we are able to apply the equation to all exponents. Like this:

$$nz^{n-1} = \Delta z^n$$

And we don't have to prove or derive the table either. The table is true by definition. Given the definition of integer and exponent, the table follows. The table is axiomatic number analysis of the simplest kind. In this way I have shown that the basic equation of differential calculus falls out of simple number relationships like an apple falls from a tree.

We don't need to consider any infinite series, we don't need to analyze differentials approaching zero in any strange way, we don't need to think about infinitesimals, we don't need to concern ourselves with functions, we don't need to learn weird notations with arrows pointing to zeros underneath functions, and we don't need to notate functions with parentheses and little "*f*s", as in $f(x)$. But the most important thing we can ditch is the current derivation of the magic equation, since we have no need of it. This is important, because the current derivation of the derivative is gobbledygook.

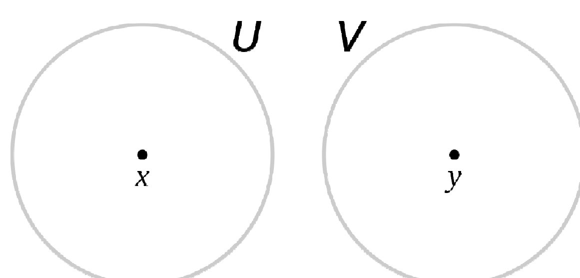
(To see the shocking reasons why, read the rest of the paper "Calculus Simplified" at www.milesmathis.com, paper #48. Also the beginning of the paper is pretty remarkable, and Mathis has many other interesting papers on physics and mathematics at his website. All works on the site www.milesmathis.com are the copyright of Miles Mathis and may be reproduced for educational, non-commercial use only. I quoted him at length above to give you an idea of how creative and different his approach is, dispensing with all the confusing limits involving contortions with *epsilons* and *deltas*, that are apparently used to cover up what looks suspiciously like division by zero. What Mathis describes regarding calculus in terms of orderly series of integers is definitely something the ancients were aware of centuries before Newton's time. We just do not know how far they applied it. Maybe someday we will discover a cache of ancient mathematical texts and be able to throw more light on the question. There are other aspects of the calculus that require other approaches than that of Mathis, such as derivatives of trigonometric functions, hyperbolic functions, and natural logarithm functions. They all may be defined without recourse to a theory of limits.)

From the example we picked above ($y = x^2$) all we need to do is to pick any point on our curve $y = x^2$. For example, suppose we pick the point on the curve $x = 3$ and $y = 9$. From the table of differentials figured out by Mathis without resorting to any *epsilons* and *deltas* we know that the constant slope of the tangent at that point will be nx^{n-1}, where $n = 2$ and $x = 3$. So 2 times 3 equals a slope of 6 for our chosen point.

The Zeno Experiment: Go outside of your house or apartment and stand at some distance from the front door. Leave the door open. (If the weather is inclement, do this exercise indoors by going from your living room to another room you can reach by walking straight forward.) Walk half way to the doorway. Stop. Now walk half of the remaining distance to the doorway. Stop. Now walk half the remaining distance to the doorway. Stop. Continue in this fashion until you find yourself standing on the threshold of the doorway right at the jamb. What prevents you from stepping inside?

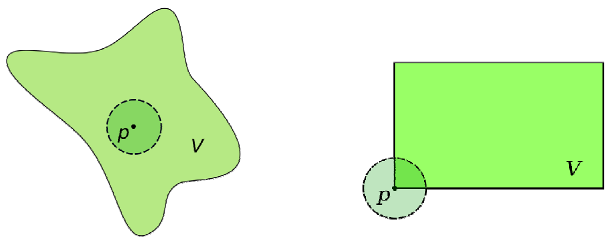
The above exercise is based on observations made by the ancient Greek thinker Zeno and demonstrates the power of challenging a generally accepted assumption (e.g., that you have to find a "limit" by means of infinitesimals) and moving in for a closer look to see what is really going on behind the smoke and mirrors.

What about the **neighborhoods** by which our limit was defined by the mathematicians? Hausdorff space is developed from two primitive **undefined** components, sets of *points* and subsets of these points called *neighborhoods*. Here is a graphic interpretation of these undefined elements.



The points *x* and *y* separated by their respective neighborhoods *U* and *V* and perhaps some other space as well. (**Wikipedia**, “Hausdorff space”)

We do not know what a "point" is. It could be an inhabited dwelling among a collection of nearby inhabited dwellings in a local geographical area. The dwellers in the dwellings may share some common lifestyle, culture, and/or a relative, but not necessarily strictly defined, distance between their dwellings that could range from high rise apartments to rural farms. The neighborhood has more than one "point", but we have no idea how many points there may be, and for size-less points there may be an infinite number of points in the neighborhood. However, there is a small restriction placed on neighborhoods.

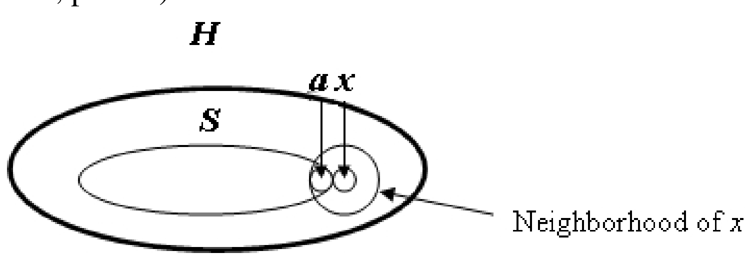


"A set *V* in the plane is a neighborhood of a point *P* if a small disk around *P* is contained in *V*. A rectangle is not a neighborhood of any of its corners.

"Intuitively speaking, a neighborhood of a point is a set containing the point where one can move that point some amount without leaving the set." (**Wikipedia**, "Neighborhood") Note the wiggle room in the neighborhood and the fuzzy quality of the definition of a neighborhood. Like our suburban neighborhoods it can be subject to sprawl.

In Hausdorff space the "limit point" is defined as follows:

* "A point *x* of *H* will be called a limit point of a subset *S* of *H* provided every neighborhood of *x* contains at least one point of *S* distinct from *x*." (Eves and Newsom, **Introduction to the Foundations and Fundamental Concepts of Mathematics**, p. 260.)

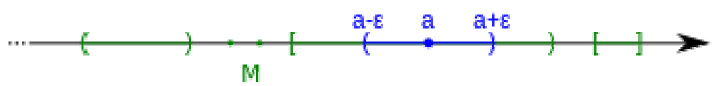


The point *a* is in *S* and is also in a neighborhood of point *x*, and point *x* does not have to be in *S*, but the neighborhood of *x* must have at least one point that is not *x* but is a member of *S*. Point *a* is an example of such a point in *S* that is distinct from x but in a neighborhood of *x*.

From this definition of a limit one may derive the theorem that a neighborhood contains an infinite number of points. But when we look closer, we find that the concept of *neighborhood* itself is defined in a somewhat vague manner. This is a fundamental characteristic of **all** mathematical systems, however marvelous and useful they are. When you go down to the bottom of any mathematical system to see what it is made of, you find that it is built from at least TWO **undefined** elements – in this case, **point** and **neighborhood**. We can add the observer as a third member of the triumvirate of undefined entities. The house of cards is built on uncertainty. Uncertainty arises from undefined or only partially defined entities. The nature of an undefined entity is worth exploring, and we will do that in some detail as we explore observer physics.

When we go down there to find the limit and we get into the "neighborhood" of the limit, suddenly everything gets a bit fuzzy. The fuzziness comes from the uncertainty that is built into mathematics at the ground floor -- just like in physics. In the system of real numbers this uncertainty shows up most clearly in the irrationals.

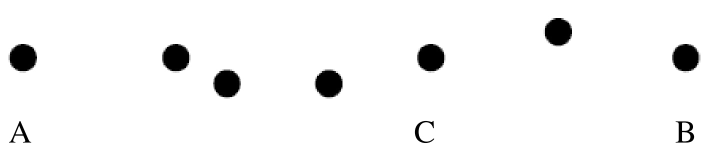
The non-algebraic irrationals (transcendental numbers) make up the major "spatial portion" of the numbers between 0 and 1 on the Real Line Segment.



The set **M** is a neighborhood of the number **a**, because there is an **ε**-neighborhood of **a** which is a subset of **M**. The **ε**-neighborhood includes any real number (point) that is a distance less than **ε** from **a**.

If mathematical points are size-less (e.g., point **a**), the irrationals must therefore HAVE FINITE "GAP" SIZE (the **ε**-neighborhood, exclusive of but surrounding **a**). Otherwise you will never get anywhere in filling the space between two "points" no matter how many infinite buckets of points you throw in. The gap size is indeterminate. There is no standard for it. Lines are made of points and certain subsets of points called neighborhoods. I propose that we let the points be the rationals and the neighborhoods be the irrationals. How does that work?

The observer defines the SIZE of a neighborhood.



Exercise: Here is a simple exercise to help you understand this principle.
(1) Mark two dots A and B an arbitrary distance apart on a blank sheet of paper.
(2) Add dot C anywhere between the two starter dots.
(3) Add more dots anywhere between A and C or C and B.
(4) Continue this process until you begin to see a line manifest between the two starting points.
(5) All the dots represent point-value integers, rationals, and/or algebraic numbers.
(6) The spaces in between represent "irrational" neighborhoods of no defined value, although we can define a value for them by giving precise values to the dots.

Potentially the two original points A and B determine a line, but the line is not yet manifest. Your mind fills in the line as straight or curved, depending on how you place the intermediate points. The dots become a line only when you **believe** they become a line, and not until then. This is **observer physics**. The observer controls the whole process and determines how and when the "quantum leap" is taken from dots to line. That leap is a leap of faith. The dots (particles) become a line (wave) only **because you believe they form a line**, and only **when** you believe they form a line. With practice you can shift your viewpoint and see the line once again as a set of dots. This corresponds to the quantum process of collapsing the wave function.

So the process of the "quantum leap" is **reversible** and thus **symmetrical**. In fact we have just done the reverse of the usual quantum leap in physics that occurs when a continuous wave “jumps” to form a dot-like particle somewhere in the wave function when the observer observes it!! The reverse of the quantum leap from wave function to particle is the creative process in which one imaginatively connects unrelated items into a composite whole. The quantum "collapse" is caused by directing the attention deliberately or non-deliberately from the potential to the actual state of experience. The non-deliberate process may be due to habit, a sudden stimulus, persuasion (indoctrination of some sort), and so on. Deliberate direction of attention is a function of the will and skill in tuning and directing the attention.

We assume that the mathematical dots can be infinitesimally small. But this is not true in the "real" world. Even thoughts occupy mental space and require energy. We are stuck with the situation that at some subjective "point" in our experience of the process our mind fills in the line from the stand point of perception, like a TV image getting smoothed out. Instead of dots we see a line. Do the steps outlined above and notice when the dots seem to connect in your mind to give the impression of a line. This is the game of "Connect the Dots." Can you "disconnect" the dots? Practice connecting and disconnecting the dots until you can go either direction effortlessly. This kindergarten exercise is very profound science and may condition the mind to imaginative creativity as well as understanding quantum physics.

If the dots are finite, then we don't need "neighborhoods" to fill the gaps. We just fill the space with dots and count them if we wish. If the dots are size-less, then we need "neighborhoods" to fill the gaps. But as long as we can distinguish one dot from another, there must be a gap between the dots.

This is the principle of the **GAP.**

In his course called **The Science of Creative Intelligence**, Maharishi emphasizes the notion of a gap that sits between any two creations. Palmer also discusses this principle in his **Avatar Materials**. The notion of a gap is essential to the notion of **continuity** in the physical world and also in the mental world. **Gaps** and **continuities** are codependent. We can imagine a finer density of dots than we can perceive with our naked eye. But, if we try to imagine an infinite density of dots, our intellect fuzzes out and makes a quantum leap to **imagine** a continuity. Thoughts have a finite quantum resolution just as our eyesight. (Do the above experiment in your mind's eye, filling in dots mentally until they become so numerous that you are unable to distinguish separate dots.) **Dots and gaps are complementary.** The gap has indeterminate size. It is undefined unless defined by you or someone to whom you delegate the task. The "ungaps" -- or dots -- on the other hand, can be well defined (position-wise), but need not have any size. Or you can switch them around. Generally we stick in an arbitrary amount of gap numbers to fill in the gaps and complete the intended structure.

Cantor liked to think of the set of irrationals (gaps) as hugely infinite, "uncountable", and infinitely more numerous than the infinity of natural numbers (dots). We can just as easily envision a gap of some finite size between each dot. However many dots we have, finite or infinite, determines the number of gaps. The value of a dot "labels" the adjacent neighborhood gap peanut number, giving it a precise "ordinal" value. We can label a dot as (*Na*), and its partner gap as (*Nb*). For every (*Na*) there is a corresponding (*Nb*), where *a* and *b* are natural number indexes. Thus we find exactly as many gaps as dots, minus one, the terminal point, which has no partner.

You decide how many dots and gaps there are. Minimum: 2 dots and one gap. Maximum: "infinity" dots and infinity minus one gaps. There's always one more dot than gap for a line. You need a terminal at each end of a line segment. See how simple it is? That terminal dot can serve as the "limit" to a finite or infinite series of dots.

However, one thing is certain. Gap numbers are fundamentally different from dot numbers, just like qwiffs and particles are different animals. Yet they are inseparable like the two sides of a coin. They have a codependent relationship. Gap numbers are "connect-the-dot" numbers. They can't exist without dots. They are dot-dependent. They are countable because each gap is labeled by its contiguous dot that defines it. If the dots are finite, the gaps are finite. If the dots are infinitely many "mathematical dots", the gaps are infinitely many "mathematical gaps". The gaps have no fixed size. The observer determines their sizes. This is his metric, and he may make each gap the same finite size if he prefers.

Cantor formulated a clever "proof" that the real numbers are uncountable. We'll look at it in more detail later (Chapter 3). In framing this proof Cantor turned up a paradoxical problem in mathematics (akin to Gödel's incompleteness theorem), because the purpose of **numbers** is to **count**, so one wonders what use uncountable numbers might have unless they are made of some other "hyperdimensional" material that has some use in some future infinite dimensional space. Cantor's proof by contradiction begins by supposing a theoretical list of **all** the real numbers between 0 and 1. A list is by definition a series of items written, printed, or imagined one after the other, and thus is countable. Cantor imagined his list. Then Cantor "diagonalized" his list, changing digits in such a way as to generate a new number from the list but not on the list. Thus he claimed to have shown the "un-countability" of the set of real numbers.

It seems very strange to say that irrational real numbers are a kind of **number** you can't

count. The uncountability is due to there being endless numbers of non-algorithmic irrational numbers with no precise sequence of digits. How do you know which is bigger and which is smaller if you can't write them in full or describe a method for writing even a single one? You can't count them in a sequence as a set or even write a single one -- and yet you apparently need them to do math! This smacks of contradiction. Such a situation is like proving that yes is no or that 1 = 0. Numbers are supposed to have values that represent precise quantities and/or precise positions in an orderly sequence.

The interpretation of the non-algebraic irrationals that I propose -- gap numbers or entire "neighborhoods" -- posits them as REAL real numbers. In my interpretation they map one-to-one (minus one if you include both terminals) to the natural numbers or rationals, are countable, and behave like ordinary numbers. But, like imaginary numbers, they are in a different mental dimension of reality from the ordinary numbers. You can not write any one out except by its dot-label (unless you assign a specific metric to it or all of the gap numbers, but each one is always at least labeled by the dot number next to it. The two different mental dimensions, rational and irrational, are nicely connected, complementary, in a one-to-one relationship, and you can jump from one dimension to the other with no problem. Continuity becomes a non-issue. The role of the Observer becomes a major issue. The Observer optionally may define the resolution of the system, by setting the size of the neighborhoods as an arbitrary standard metric distance between points or by an algorithm. The threshold of continuity, the stage at which the collection of points quantum leaps to become a continuous line -- or, vice versa, -- the continuous line quantum leaps into a collection of dots remains in the Eye of the Observer.

Redo the "Connect the Dots" exercise and pay close attention to the moment when you suddenly see the set of dots as a line. Does the threshold change when you do the exercise more than once?

The old child's game of "connecting the dots" turns out to be a fundamental technology of quantum physics. The dots are particles, and the finished drawing is the quantum wave form. Once we understand this game more deeply, we will understand the transduction of energy, (how the forces fit together), and the vital role of the Observer in the whole scheme.

The observer makes decisions and interpretations, all of which are judgmental processes that introduce bias into a system.

Any element of a system may have several, or even unlimited, different interpretations. We may interpret an element as a graphic element, a symbolic semantic element, a numeric element, a relationship link, a transformational operator, and so on. How an element is used is an arbitrary choice of the scientist who is organizing and applying the system. Some believe that the numerical interpretation of elements -- counting ability -- may even qualify as a design feature of human language. I suspect that it is merely another name for the feature of productivity and the semantic duality of assigning meanings and interpretations. This also may be completely arbitrary.

Question: If you examine the data in a computer, you find it consists of nothing but arrays of numbers in the form of 0's and 1's. Yet some of these 0's and 1's represent numbers, some represent words and messages, some represent graphics, and some represent program instructions. How does the computer know which is which? Who decides that?

Let us now back up and see if we can set up some definitions at the foundations of number theory so we understand logically what in the (mental or physical) world we are talking about. (This section may seem abstract and repetitious, because it is!!)

What is a **number**? **Numbers** are now usually defined in terms of **sets** and certain possible properties of sets described as **cardinality** and **ordinality**. **A set is a countable collection of definite and distinct similar objects called elements or members.** Cardinality refers to a certain quality of a set, and ordinality refers to the relation of one set to another or of one element to another within a set. Cardinality is the answer to the question, "how many?" It defines how many elements are in a given set. Ordinality is a linking of **numbered** sets (sets containing various amounts of members that have been **counted**) by a relation such as "is less than" (<) or "is greater than" (>). It can also refer to the linking of elements within a set by the process of one-by-one **counting**.

A cardinal **number** is a label placed on a set. The number label indicates uniquely how many members there are in the set, rather than any other properties assigned to the members. Cardinality is an expression of the linguistic design feature of discreteness, and ordinality is an expression of the design feature of productivity. Does ordinality also contain an idea of discreteness? We defined **number** in terms of **countability**. How do we define **countability**? It is the ability to place labels on sets to represent the **number** of members in each set after we have **counted** all the members **one by one**. We are circular. Circularity of definitions implies the principle that at its basis every language (including mathematics) begins with undefined elements.

The **American Heritage Dictionary** defines **number** as "one of a series of symbols of unique meaning in a fixed order that can be derived by **counting**." "Symbol" is defined as "one that represents something else . . . " "Represent" is defined as "to stand for; symbolize." **Counting** is "to name or list (the units of a group or collection -- i.e., the elements of a set) **one by one in order** to determine a total." The process of **counting** -- listing **one by one** and labeling each element with a unique **number** label -- creates an order within a collection of elements, and from the ordering we determine the total **number** of elements as the conventional number label we apply to the last element in the **counting** sequence. A total is "an amount obtained by addition; a sum; a whole quantity; an entirety." In other words a **total** is the **number** (quantity, how many) you get as a result of a complete (whole, entire) **counting** of the elements of a set. **Wikipedia**, "Counting" says: **Counting** is the action of finding the **number** of elements of a finite set of objects. . . . In mathematics, the essence of **counting** a set and finding a result *n*, is that it establishes a **one to one correspondence** (or bijection) of the set with the set of

numbers {1, 2, ...,n}.) **Counting** is defined by **number**, and **number** is defined by **counting**. A name is "a word or words by which an entity is described or distinguished from others." Name as a verb is "to give a name to." In a circular definition an entity is described by words that are functionally equivalent. To **number** things is to **count** them. So what is in a name? A list is "a series of names, words, or other items written, printed, or imagined one after the other." A series is "a **number** of objects or events arranged or coming **one after the other** in succession." (A list or series apparently requires **more than one** object or event.) Order is "a condition of logical or comprehensible arrangement among the separate elements of a group." (A group also requires more than one element.) What the logic or comprehensibility is that a person uses to put a collection in order is apparently arbitrary, which means it may be randomly chosen and defined and is orderly only because someone says so. The elements within a given set need not be counted in any specific order, and this gives rise to the notion of combinatorics. As long as the elements of a given set remain unchanged and the **counting is one by one**, the result of the **counting** gives the same **number** each time it is done on the same set no matter the order in which the **counting** of elements is done, so long as it is **one by one** and the contents of the set remains unchanged.

A symbol "represents" something and "to represent" is "to symbolize". "To stand for" helpfully suggests that a **number** is a name, or symbol that is a unique substitution of one thing for another thing. The notion of coming **one after the other in succession** seems important for explaining what a series is and giving the notion of order. When we **count**, it is helpful if each element of a group of separate elements has a symbolic name (i.e., its **number** label), and the names must be defined as arranged in a fixed order by means of a logical or otherwise comprehensible arrangement.

An **element** is an object of our perception or thought that has become "a fundamental, essential, or irreducible constituent of a composite entity; a member of a set." A **set** is "a group of things of the same kind that belong together and are so used. . . . ; a collection of distinct elements having specific common properties." A **member** is "a distinct part of a whole." **Member** and **element** can be used interchangeably when referring to **sets**, but member refers to the way an element belongs to a group, and element refers to the member's essential irreducible quality. From these statements we gather that **sets must contain two or more elements**. A set consisting of **one element** is not really a set. One is not a group, and you can not count one element **one after another** or **one by one**. These definitions as well as the notion of **order** require at least **two elements** in a set. If a set only has one element, that element that is a member can not be a **distinct part** of a whole, because it **IS** the whole. All of this reveals that we have been counting elements one by one with no notion of what one is. We have assumed a number without first demonstrating its existence or even defining it as a postulate. Giuseppe Peano realized this when he created his axioms of arithmetic, so he included a postulate asserting the existence of a natural number 1 such that $a \times 1 = a$ for all a in N. Here a is any natural number in the set N of natural numbers and "×1" is the identity relation with multiplication. Later they added $a + 0 = a$ for all a in N as the identity relation for addition.

An element is a member of a set, and a set is a collection of elements. For some reason we **believe** that those elements belong together. And at some point we just have to **believe** that we know what we are doing when we are **counting** with **numbers**. Despite the circularity of the definitions, the concept of a set with its elements is helpful, and modern mathematicians since the time of Cantor have developed a strong emphasis on set theory as a way of explaining the foundations of mathematics.

To "set up" set theory as a system for generating the idea of numbers, it is usual to begin with a few fundamental definitions. Here are three start-up definitions from an introduction to sets based on Eves and Newsom, pp 249-251. [I should mention that Eves and Newsom generally is an excellent book on the history and foundations of mathematics. Yet they get muddled when it comes to defining set theory.] (1) Two sets are in **one-to-one correspondence** if we can pair the elements of each set so that each member of each set corresponds to one and only one member of the other set. (2) Two sets *A* and *B* are **equivalent** ($A \sim B$) if and only if they can be placed in one-to-one correspondence. (3) Two **equivalent sets have the same cardinal number**. [We already noticed that we have been doing an operation with the number one before we have generated the idea of what a number is, much less what "one" is. You will also notice that the number "two" is used in all three definitions.]

Our **counting numbers** are often then (based on definitions like the above) defined in terms of the notion of the cardinality of a set: for example, we can say that $\{a\}$ represents cardinality of 1; $\{a, a'\}$ represents cardinality 2; $\{a, a', a''\}$ represents cardinality 3, and so on. This generates what we can call the "tally" numbers whereby we simply represent a number by a group of arbitrary similar objects (tallies) that are placed within a "set", that is, defined as a single group or collection. The use of the superscript apostrophes also forms a group of similar tally objects, and the set $\{a\}$ that represents cardinality of 1 lacks an apostrophe. So we can say that in terms of apostrophes, the "start-up" set in this set of tallies represents an "empty apostrophe set". Now here is something to wrap your mind around, and I will quote it as definition 4 is presented in Eves and Newsom, p. 251:

* (4) A nonempty set is said to be finite if and only if its cardinal number is one of the cardinal numbers 1, 2, 3, ... A set which is not empty or finite is said to be *infinite*.

Notice that definition (4) starts out talking about a "nonempty set" when the definition of a set has been previously given by Eves and Newsom and elsewhere as "a collection of definite distinct objects of our perception or thought," and the objects in the collection are to be called the *elements* of the set. So how did the notion of an "empty" set with **no elements** get slipped in? How can *nothing* be considered a definite distinct object of perception or thought? What is an empty set? With no prior knowledge of a definite and distinct object that can be an element in a particular set, how can we say what it means to have an empty set? Oddly, the notation used here to describe the generation of our counting numbers in terms of cardinality uses a "double" tally system with two symbols *a* and *'*. The set $\{a\}$ in terms of the ' symbols represents an empty set. However, the way this notation is set up, we do not have a one-to-one mapping of *a* and *'*.

For one-to-one correspondence we have to have $\{a\} \leftrightarrow \{'\}$, $\{aa\} \leftrightarrow \{' \ '\}$, $\{aaa\} \leftrightarrow \{' \ ' \ '\}$, ... Then we know that the two sets are equivalent, and that set {a} is not really an empty set. On the other hand, we now have a notion of an **infinite** set as one whose cardinality is undefined in another way with respect to "how many". It is not empty, and it has so many members that they are uncountable within a **finite** process of counting. And all of this occurs by rampantly using the notions of one and two in order to define what one and two are along with an unspecified "number" of other "natural numbers".

Some mathematicians like to derive number theory from set theory, by beginning with the notion of an empty or null set ∅. Then they postulate the existence of a "successor" to ∅. In both cases these must remain undefined to avoid circularity. How can we derive all sets with various members from an empty set when the notion of a set implies that something has to be in it? If you remove all of its members, is a set still a set?

Also nowadays some mathematicians pretend to play with sets that contain **uncountably many** members as if they can distinguish one form of uncountability from another. What does it mean to be "uncountable" (or "non-denumerable")? This idea certainly seems to spring from the notion of infinity, or maybe it is just playing with prefixes to generate nonsense.

The concept of a set is a **transcendental, metalinguistic construction** we use to organize objects into groups of elements. "Transcendental" means that the organizer is not a member of the elements that he/she organizes. "Metalinguistic" means to use a language to talk about the same or another language. **Set** and **element** are codependent notions when used in this manner but exist at different levels of discourse. To have a **set** we must have an **element**, and an **element** in the sense we are using here **must be a member of a set**. Only in this way can a set have **at least one definite and distinct element (and if a set implies a collection, it must have two or more elements)**. The notion of "Nothing" seems undefined and indistinct and therefore may not qualify as an element of a set. If zero is to be a member of a set, it must be defined and distinct in some way and related by some property to the other members of the set. If we simply make 0 an element in a set with other symbol elements: {f, 7, 5, m, 0, &, @}, then none of the elements in this set has any mathematical value. These are simply objects that we have grouped together, perhaps because each consists of a single symbolic mark of unknown meaning. We can put them in some numerical order mapped one-to-one to the counting numbers: 1 = f, 2 = 7, 3 = 5, 4 = m, 5 = 0, 6 = &, and 7 = @. From the metalinguistic viewpoint of the set the symbol 0 has no mathematical meaning other than as an element of this particular set. We can reorganize the set so that 0 corresponds to any **number** between 1 and 7, but that only tells us an order we have imposed on the set.

If zero (0, an empty subset, ∅) is a **member** of a set, it is hard to determine what part of the whole it is or what property it has in common with the other members of the set other than as a meaningless object. If we treat 0 as an element, it is a member of a set and not a mathematical object. If we use the arithmetic operation of addition and add 0 to the total of a collection of other integers, it does not change the total in any distinct way. It is "an element of a set that when added to any other element in the set produces a sum identical with the element to which it is added." Thus it must be treated as an ordinal number indicating an initial point or origin that comes before the first element in the ordering of the elements. However, if it precedes the first element, it does not possess the common properties of the elements that will follow it since by definition it lacks them all. As a cardinal number 0 indicates "the absence of any or all units under consideration." A cardinal number is a number "used in counting to indicate quantity but not order". Zero can be said to indicate order, but it does not indicate a quantity; it indicates a **lack of quantity**, and thus may be considered unsuitable as a cardinal number.

Graphics connect with sensory images by an imagined resemblance to prior experience. Linkage appears as juxtaposition in time or space or other dimensions. Operators suggest processes of physical transformation and change relative to a prior condition. Symbols, by conventional agreement, represent personal choices and interpretations. Cardinal numbers express an experience of sets of objects that are discrete, yet share common properties within the set. The number label represents the notion of how many members are in the set. Those members of the set often can be counted in various ways, but always give the same number answer to "how many". We might imagine a set containing one member (**despite the need for at least two members to form a group**) and that member is characterized by the absence of all possible units under consideration, but, if you do not priorly know what units are under consideration in any such element, can that member be joined in such a set by any second member that can be distinguished from that member which represents the absence of all units under consideration? We have a contradiction unless we imagine a set that may consist of elements such as no apples, no oranges, no bananas, no Volkswagens, no windmills, no petunias, no whales, and so on. Since we do not know what other elements might represent a lack of some unknown things, we basically have one set that represents the Vacuum State of the Universe in the form of all possibilities but with none available for experience. The number of "elements" in such a set consisting of null elements is indeterminate and thus undefined. So it does not seem possible to derive all the numbers from an empty set that indicates the absence of any or all units under consideration. How does one extract a set containing something from the undefined elements of a Vacuum Set other than simply to announce that one has extracted such a set by a quantum leap of faith? A set with cardinality zero can only function in this way by magical creation. You simply say, "Look! Here is something! I call it an apple." This is not the logical language of mathematics, it is magic.

Quine puts forward this ontological problem at the beginning of his book, **From a Logical Point of View**: "Nonbeing must in a sense be, otherwise what is it that there is not?" This is a chicken-and-egg problem in a universe where the existence of things as physical phenomena is transient. The nonexistence of something in the present does not guarantee its nonexistence at all times and places in the universe. On the other hand, how do you justify the reality of concepts that have no experiential reality, numbers that can never be expressed in any way other than by assigning an arbitrary symbol to them and saying "there is a number that exists" but with no way to identify it precisely or to construct it. It becomes a matter of belief, and the transfinite mathematics that has proliferated is more like a cult than a scientific discipline.

From a mathematical viewpoint we may consider "ontology" more a matter of logical consistency. Illogical statements can be imagined and communicated, but they do not "exist" in the world of mathematics, because **mathematics is use of language in a logical manner**. When a physical phenomenon is shown to exist by experience or experiment the ontological question becomes: How do we explain it in a logically consistent way? For example, if we find that particles demonstrably have mass, a mathematical field theory that provides no reason to introduce an elementary quality called mass is either wrong or insufficiently developed. The so-called Higgs particle that some physicists feel they have found is possibly just a reasonably expected higher energy resonance of lower energy particles and does not qualify as a Higgs particle that generates the mass of quantum particles, **including itself**, unless this assertion can be tested experimentally and resolve all issues pertaining to the claims. At higher energy levels other resonances will appear, and that is no explanation of where mass comes from. (In later discussions I will propose a viewpoint from which mass becomes a natural component of the physical world we experience and has a simple origin.)

Perhaps the closest we can come to an "empty set" is with a set that contains, for example, plus 5 apples and minus 5 apples -- in other words, a set with elements operated on by an operation of arithmetic that cancels the cardinal value of the set. That would give us a set that contains transient elements or that changes into another set. For example, we may have operators that make apples appear and then disappear. For this reason I propose that it only makes sense to introduce zero as a member of the set of **integers**. Once we can define 1 as 6 - 5, or 3 - 2, we can define 0 as 5 - 5 or 7 - 7, and so on -- something you are unable to do without the operations **plus** and **minus** of arithmetic. Zero then becomes meaningful as the result of some operation that removes a given element or group of elements from within a set and turns those actual elements into potential elements. For example, suppose a fruit market has a sale and sells out all its cherries, pears, and oranges. The market then has an empty "set" consisting of three empty fruit bin subsets -- bins for cherries, pears, and oranges. We will not worry about exactly how many members were originally in each subset, but we know that the emptiness of the bins depends on the prior presence of some number of the specified fruits in each bin plus an operation that removed them. We are dealing with integers, not merely natural numbers. When we get to sets such as the **integers**, we must be careful to realize that the elements such as 0, -5, -38945, and so on must be treated simply as elements during the enumeration of the set's members. The so-called positive numbers, negative numbers, and zero in the set of integers are to be treated as lower level elements in the set with no specific numerical value other than their inherent quality of numericity and individual uniqueness in that we may enumerate them for purposes of determining the cardinality of the set. They can be counted in any order. The mathematical meaningfulness of 0 only occurs when the operations on the set of integers occur, and there are further problems when 0 is incorporated into operations on the set of rationals, the set of reals, and the complex numbers. It would seem that some other way of treating the notion of an absence of numerical value would be helpful, since problems with 0 create difficulties in many areas of physics as well.

Here is another fundamental problem with the notion of a set that occurs if we let its definition become too general so that it bleeds over into the definition of the set's codependent partner, the element. For example, suppose we let X represent *any* set, and we **let N represent the set of all sets that are not members of themselves**. Then, from our definition of N,

* $(X \notin X) \leftrightarrow (X \in N)$. [If X is not a member of itself, X is a member of N.]

If we allow X to be the set N, then we get a contradiction:

* $(N \notin N) \leftrightarrow (N \in N)$. [If N is not a member of N, N is a member of N.]

And of course that means we generate paradoxical nonsense at the very foundation of the notion of a set if we are allowed to think of a set as able to be an element within a set. Sets and elements exist in different levels of discourse, and there are rules by which they can interact logically. Elements can interact mathematically via operators within their own discourse, but from the level of sets, they are merely elements and are treated differently. Therefore, it is not possible by our rules of discourse for a set to be a member of itself and this famous paradox simply disappears, being recognized as a misunderstanding of the relationship between a set and an element.

Having discovered that the set theory they were developing was riddled with contradictions at its very foundations (e.g., Russell's paradox, Burali-Forti paradox, Richards' paradox, etc.), the transfinite mathematicians beat a hasty retreat and totally revised their approach to set theory. In that process they so completely redefined sets that "modern rigorous" set theory has no connection with the "naive" version that went before. In modern set theory we again have the notion of **set** and the notion of something variously called an **element**, **member**, or **object.**

The basic assumption is that "all mathematics is translatable into logic." (Quine, p. 80), referring to the **Principia** of Whitehead and Russell as "good evidence" of that possibility. We will begin again in a more modern approach, this time with "basic concepts and notation" from **Wikipedia**, "Set Theory").

* **Set theory begins with a fundamental binary relation between an object *o* and a set *A*.**

(This new start-up begins with **two** names that are completely **undefined symbols: *o* and *A***. Set A and object o relate in a "binary" manner, binary meaning that we have **two** things interacting, a set and an object. The word "fundamental" does not cover over this glaring numericity "binary" that precedes a demonstration of numericity.)

* If o is a **member** (or **element**) of A, write $o \in A$. Since sets are objects, the membership relation can relate sets as well.

(The objects in the binary relation are given two possible names, a **member** or **element**, but each of these remains undefined, although we get a feeling that a member "belongs" to a set, and that it is an elementary object. Thus far we have an **undefined** relation (symbolized by $\in$), and **sets** are also being called **objects (which dilutes the hierarchical distinction between sets and elements)**, so that there is now no distinction between them. Sets and elements can be objects. Therefore, the expression $o \in A$ implies that an object can be an element/member of an object, which is nonsense unless we make it clear that these are two different objects or perhaps the intended relation is an equivalence relation. Since "element" is also undefined, we essentially have discovered that "da doo da." We could as well say $A \in o$. (A set is an element of an object.) So far this new presentation seems already in trouble.)

* A derived binary relation between two sets is the **subset relation**, also called **set inclusion**. If all the members of set *A* are also members of set *B*, then *A* is a subset of *B*, denoted $A \subseteq B$. For example, {1, 2} is a subset of {1, 2, 3}, and so is {2} but {1, 4} is not. From this definition, it is clear that a set is a subset of itself; for cases where one wishes to rule this out, the term **proper subset** is defined. *A* is called a **proper subset** of *B* if and only if *A* is a subset of *B*, but *B* is **not** a subset of *A*."

It appears we must be very clear that **sets** and their **elements** are quite distinct but codependent entities operating on different mathematical (linguistic) levels. The notion of a set is simply a way of grouping elements, so a set should not be an element of a set. We may identify "proper subsets" within a set so long as their number of elements is less than that of the set. A subset can be treated as an element. We find then that if we think of sets and their elements in terms of the symbols we use to discuss them, then the notion of a "set" is a **meta-symbol** for speaking of groups of lower level symbols that represent elements as the members of the "set". These two levels of discourse are linked together but always live in different dimensions. This distinction gets blurred when ideas like **set inclusion** appear. We will see as we explore Observer Physics that confusion occurs in physics when dimensions are not properly defined. A dimension must have a definite and distinct property, and we must be careful not to mix dimensions with different properties or to separate entities of the same dimension as if they belonged to different dimensions. The notion that a set can be included in another larger set opens up the door to confusion.

Mathematicians have studied the types of numbers that can exist under the ordinary arithmetic operations. The most primitive numbers are called the natural numbers, $\mathbb{N}$. They are what we call the whole numbers. They are the set of positive numbers related by the "size" dyadic relations, ($a < b$) and ($b > a$) and ($a = b$). Then come the integers, $\mathbb{Z}$. They are a set related by the binary operators ($a + b$) and ($a - b$) as well as the size relations. Then come the rational numbers, $\mathbb{Q}$. They further include the binary operators ($a \times b$) and (a / b). Then come the real numbers that include irrationals, $\mathbb{R}$, and the imaginary numbers, $\mathbb{I}$ (called complex numbers $\mathbb{C}$ when they include all the reals. They still further include the operators $[(a)^n]$ and $[(a)^{(1/n)}]$. For example, $2^{(1/2)}$ is irrational, and $-1^{(1/2)}$ is imaginary. With these sets we have sufficient operators to do all of ordinary mathematics. We also have numbers that correspond to every relation and operation defined in the system. However, we must define 1 at the axiomatic level.

We also have problems with zero in all these sets except possibly the integers, ℤ. And if there are problems elsewhere, we should take a hard look at the way we use 0 and contemplate why the ancients had no concept of 0 and why the Egyptians put so much importance on unitary fractions (rational numbers in which the numerator is 1 and the denominator is some whole number greater than 1.)

Modern mathematicians have developed a very convenient notation for representing numbers with great precision: the radix point positional system. By using a radix point dot to divide the digits of a number into two groups, one can write integers to the left of the dot and fractions to the right of the dot in the positional system. This is a grammatical device in which syntax is indicated by means of the order of the elements. (For example, English and Chinese grammars have strongly positional syntax.) The portion of a number representing a whole value in the chosen base becomes the sum of one or more whole numbers with less than the base value times the base to some positive power (including zero). The portion of a number representing some fraction of unity is expressed in any base as the sum of a series of fractions each of which consists of a whole number numerator with less than the base value times a unitary fraction of the base to some negative power. For example, in the decimal system (base 10), the number 463.72 expresses the sum ($\mathbf{4}\times10^2$) + ($\mathbf{6}\times10^1$) + ($\mathbf{3}\times10^0$) + (7×10^{-1}) + (2×10^{-2}), noting that $10^0 = 1$. (Any number x [except 0] to the zeroth power is 1, because $x^{(n-n)} = x^n / x^n = 1$. In the case of 0^0 the result is indeterminate, because 0 times itself any number of times is still 0, and division by 0 is indeterminate. See further discussion of 0^0 later on.) The base-2 number 10.011 denotes the sum $1\times2^1 + 0\times2^0 + 0\times2^{-1} + 1\times2^{-2} + 1\times2^{-3}$ and corresponds to the base 10 number 2.375." At present there is no universal standard for how to write the positional radix point. Some countries use a period (.), some use a medial dot (·), and some use a comma (,). We use a period in this discussion.

One problem with this notation system is that **no matter what base you choose**, many rational and irrational numbers actually produce infinitely long sums of fractions. A vinculum, dot, or arc can be placed over a sequence of digits that repeats endlessly (a repedend). For endless irrational sequences of digits a sample plus an ellipsis (...) is used. For example (and I use an underline in place of a vinculum), 1/3 = $0.\underline{3}$, and 1/7 = $0.\underline{142857}$, and π = 3.14159..., and so on. There is no way to indicate the exact value of an irrational number in the radix point format. There apparently are more irrational numbers than rational numbers, and that leads to the question of whether irrationals even qualify as numbers. Because you can not write out their exact value even in principle, you apparently can not even count them. As numbers they are uncountable in any ordinal sequence. This creates an awkward situation. Numbers originally were defined and intuitively understood as representing the countable nature of objects. Sets were supposed to be countable. But here we have numbers that no one can count or even put in any precise order no matter how hard they try! They are uncountable in principle according to this system of defining numbers.

Are these numbers really numbers? Well, there is a whole class of physical objects defined in natural human language that are inherently uncountable. Maybe there is a connection between uncountable objects and uncountable numbers.

Most natural languages have words for objects that are countable and for objects that are not considered countable. "Uncountable" objects include common items such as water, milk, butter, mud, cement, air, and so on. For example, "table" is a countable noun, and "milk" is a non-countable noun. We can have five tables, but not five milks. We can have five "cartons" of milk, because the carton is a container for the weakly bounded object, "milk." We can have five cases of cartons of milk. The cartons are subsets in the cases, and the cases can also form subsets in the whole milk shipment. But milk itself has no definite elementary unit. A single "molecule" of milk is not possible because milk is a colloid of particles suspended in water and thus lacks the usual properties we associate with milk. So cardinality of sets has something to do with the boundary that defines an object and gives it discreteness and distinctness. Usually, but not necessarily, this definition is in terms of time and/or space.

Mathematicians discovered that the algebraic description of the world of numbers corresponds exactly to a description of the world of mathematical shapes using geometry. The only difference is one of interpretation and the medium of expression. Thus was born analytic geometry and the correspondence of numbers with points on lines and in space. The invention by Descartes of a grid for calculating shapes was a major step forward in the development of modern science. His idea derived from the ancient practice of using grids to scale drawings for architectural projects and art works.

With the advent of analytic geometry, every algebraic expression could be translated into a corresponding expression as a graphic image. The theory of functions was born. The role of the uncountable irrational numbers in the geometry interpretation was that they provided continuity between the discrete natural numbers, integers, and rational numbers. It was a very challenging proposition to prove the continuum hypothesis: that the disorderly infinite sums of fractions of unity form a continuum just like a line appears to form a continuum. Mathematicians such as Dedekind felt they had achieved this, although Gödel subsequently showed that the "proof" of continuity of the real number system is inherently a postulate independent of the rest of the postulates of the real number set (rather like the parallel postulate in geometry). In other words, you can take it or leave it, just like your choice of geometry can be plane geometry, spherical geometry, hyperbolic geometry, and so on. Gödel's finding suggests that the real numbers are countable or uncountable depending on how you look at them. Gödel also apparently showed that the real number postulate system is incomplete, and it is not possible to prove even the natural number system to be consistent by methods belonging to the system itself. The first incompleteness theorem states that no consistent system of axioms whose theorems can be listed by an "effective procedure" (e.g., a computer program, but it could be any sort of algorithm) is capable of proving all truths about the relations of the natural numbers (arithmetic). For any such system, there will always be statements about the natural numbers that are true, but that are unprovable within the system. The second incompleteness theorem, an extension of the first, shows that such a system cannot demonstrate its own consistency (**Wikipedia**, “Gödel’s incompleteness theorems”).

Notwithstanding these somewhat disillusioning discoveries, the real number set still provides a nice analogy to the points on a line. It also fits pretty well with the general intuitive notion of continuity that we seem to observe in space-filling objects that lack self-defined boundaries. The uncountable irrational real numbers play the role in mathematics of the space, air, water, cement, and mud of our "real" world. They can be said to exist as objects, but they are "uncountable" because they have no clearly defined values. However, they may represent gaps of space labeled by adjacent countable values just like we use bottles to hold water within defined spaces. "Uncountables" may also serve for abstract notions such as "consciousness," "sleep," "sorrow," "happiness," "intelligence," and so on. If you can give it a label, but can not say exactly where it is or how much of it there is without adding other boundary definitions (such as "a bottle of ..."), then it is uncountable. It is a gap number. Just like the above notions from our "real world", a quality of consciousness fills in the gaps between our perception of discrete objects.

So we seem to have a handy mapping between mathematics and our world. The world of numbers and geometry can paint a pretty accurate, sometimes amazingly accurate, picture of our world. We explain this as a reflection of the principle that the physical world is an expressed form of a set of mental beliefs. But then there is the issue of predictability and certainty. These are key design features of science. The discovery of fundamental uncertainty in physics by Heisenberg precipitated a major crisis, a decisive blow to the belief that it was possible to describe all the parameters of a physical system with unlimited precision. Heisenberg showed that the parameters were arranged in a conjugate fashion that disallowed precision for all of them at once. You have to make a choice of what you want to look at closely.

In the world of mathematics, numbers are the objects. The natural numbers generate a set of precisely defined discrete mental objects. The ordinal value of each natural number is entirely predictable. From any given natural number, we can always generate the next number in the sequence by using the productive rules for number generation. But maybe this is just our choice. Maybe we have a transparent belief that natural numbers, or any other numbers, can be precise. Maybe you choose dot numbers to be precise and gap numbers to be fuzzy or the other way around, but not both ways at once. Maybe there is yet another way of looking at it.

Numbers can be mapped to physical world phenomena so that natural numbers and algebraic expressions describe actual objects and processes. Unfortunately as we shift into the quantum view of things, we discover that even apparently very discrete objects such as protons and electrons lack absolute precision and take on quantum uncertainty.

For example, suppose we direct a stream of electrons with consistent velocity through a tiny aperture to collide with a screen that registers each collision. Intuitively we expect the electrons all to strike the target at the same spot. In fact, we find that they are distributed randomly over the screen but statistically form a wavelike pattern called an Airy diagram.

On the other hand, we can describe mathematically the continuous time evolution of a quantum event by means of a continuous wave function. The function that describes the Airy diagram has the liquid wavy quality of vibrating air or water and forms a resonant pattern that fills space and time. This property of continuity resembles the real number set. But the certainty aspect of the wave function resembles more the natural numbers. At any arbitrary moment or position we know the precise value of the function and the shape of the wave. But we are totally unable to predict where any particular electron will strike the screen, just as we are totally unable to predict what the *n*th digit of a non-periodic real number will be. We do know that the electron will tend to fall at the more probable areas of the wave function's pattern on the screen, but we do not know for sure where.

So there is a lens-like operation that goes on between the mental world of mathematics and the physical world of phenomena that reverses the appearance of certainty. We suspect that this is caused by a property of the physical nervous system. What we call the "nervous system" is the medium connecting the mind and senses to the physical world. We may have uncovered a previously transparent feature of human belief systems. The two sides of the lens match in terms of "shape" but are reversed in terms of the assignment of certainty. This is exactly analogous to the way a visual image gets reversed spatially by passing through a lens or by reflection in a mirror. It is also analogous to the way a light field gets reversed in time when reflected in a conjugate mirror. We can propose a principle: **The observer's Mental Space has a relation to his physical World Space characterized by a reversal of certainty.**

It is not that the observer turns things spatially or temporally upside down in his mind from the way they "really" are. That may happen also, but we cannot prove it, because we only see what we see. Influenced by the transparent effect of the reversal of certainty, people look in the "wrong" place within the physical world for the stability and certainty they know and expect in the mental world as represented by eternal and precise natural numbers. They try to build a stable, logical world out of discrete physical objects based on the analogy of their mental world's precisely logical natural numbers. These natural numbers seem to fit so nicely to the objects they observe. Unfortunately, the physical objects that correspond to them are inherently non-local energy fields that precipitate into random and unpredictable but statistically distributed particle events when observed, so our mental mathematical mansion is built physically of sand and subject to the breeze and the tide. Alas, we seem to live in a physical world of entropy where randomness rules the roost. However, the wave functions of phenomena are orderly and predictable.

Paradoxically, if someone wants real stability, he must turn to the uncountable aspect of his world: the unbounded non-local flows of air, water, space, consciousness, and so on. These are described by the wave functions and exist physically only as patterns of huge statistical probability ensembles that shift about according to the time evolution of the wave function. Non-local flows exist forever as waveforms pulsing throughout the universe.

A single molecule of water does not make water. Water is the manifestation of the statistical outcome of trillions of randomly distributed molecules, but it obeys the laws of fluids. We will explore this "reality" more deeply in our subsequent discussions.

In this chapter we have considered what mathematics is, why it is useful for describing the physical world, and some of the theoretical and practical problems that mathematicians face. Another definition I propose for mathematics is that it is **the scientific exploration of belief systems.** With undefined primitive notions at the foundation of all mathematics, we discover that whatever we "prove" with mathematics or model mathematically in terms of scientific theories or their applications depends on what we believe about (how we interpret) our basic undefined assumptions. So in the next chapter we will explore in more detail what beliefs and belief systems are.

Appendix to Chapter 1: Some Additional Material for Study

A. "Today ZFC is the standard form of axiomatic set theory and as such is the most common foundation of mathematics." (**Wikipedia**, "Zermelo-Fraenkel set theory")

A Version of the Zermelo-Fraenkel Axioms (according to Mathworld)

(http://mathworld.wolfram.com/Zermelo-FraenkelAxioms.html)

The Zermelo-Fraenkel axioms are the basis for Zermelo-Fraenkel set theory. In the following (Jech 1997, p. 1), $\exists$ stands for exists, $\forall$ means for all, $\in$ stands for "is an element of," $\emptyset$ for the empty set, $\Rightarrow$ for implies, $\wedge$ for AND, $\vee$ for OR, and $\equiv$ for "is equivalent to."

1. Axiom of Extensionality: If X and Y have the same elements, then $X = Y$.

$$\forall u\,(u \in X \equiv u \in Y) \Rightarrow X = Y. \quad (1)$$

2. Axiom of the Unordered Pair: For any a and b there exists a set $\{a,b\}$ that contains exactly a and b. (also called Axiom of Pairing)

$$\forall a\,\forall b\,\exists c\,\forall x\,(x \in c \equiv (x = a \vee x = b)). \quad (2)$$

3. Axiom of Subsets: If φ is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : \varphi(u, p)\}$ that contains all those $u \in X$ that have the property φ. (also called Axiom of Separation or Axiom of Comprehension)

$$\forall X\,\forall p\,\exists Y\,\forall u\,(u \in Y \equiv (u \in X \wedge \varphi(u, p))). \quad (3)$$

4. Axiom of the Sum Set: For any X there exists a set $Y = \bigcup X$, the union of all elements of X. (also called Axiom of Union)

$$\forall X\,\exists Y\,\forall u\,(u \in Y \equiv \exists z\,(z \in X \wedge u \in z)). \quad (4)$$

5. Axiom of the Power Set: For any X there exists a set $Y = P(X)$, the set of all subsets of X.

$$\forall X\,\exists Y\,\forall u\,(u \in Y \equiv u \subseteq X). \quad (5)$$

6. Axiom of Infinity: There exists an infinite set.

$$\exists S\,[\emptyset \in S \wedge (\forall x \in S)\,[x \bigcup \{x\} \in S]]. \quad (6)$$

7. Axiom of Replacement: If F is a function, then for any X there exists a set $Y = F[X] = \{F(x) : x \in X\}$.

$$\forall x\,\forall y\,\forall z\,[\varphi(x, y, p) \wedge \varphi(x, z, p) \Rightarrow y = z]$$
$$\Rightarrow \forall X\,\exists Y\,\forall y\,[y \in Y \equiv (\exists x \in X)\,\varphi(x, y, p)]. \quad (7)$$

8. Axiom of Foundation: Every nonempty set has an $\in$-minimal element. (also called Axiom of Regularity)

$$\forall S\,[S \neq \emptyset \Rightarrow (\exists x \in S)\,S \bigcap x = \emptyset]. \quad (8)$$

9. Axiom of Choice: Every family of nonempty sets has a choice function.

$$\forall x \in a\,\exists A\,(x, y) \Rightarrow \exists y\,\forall x \in a\,A\,(x, y\,(x)). \quad (9)$$

The system of axioms 1-8 is called Zermelo-Fraenkel set theory, denoted "ZF." The system of axioms 1-8 minus the axiom of replacement (i.e., axioms 1-6 plus 8) is called Zermelo set theory, denoted "Z." The set of axioms 1-9 with the axiom of choice is usually denoted "ZFC."

Unfortunately, there seems to be some disagreement in the literature about just what axioms constitute "Zermelo set theory." Mendelson (1997) does *not* include the axioms of choice or foundation in Zermelo set

theory, but does include the axiom of replacement. Enderton (1977) includes the axioms of choice and foundation, but does *not* include the axiom of replacement. Itô includes an Axiom of the empty set, which can be gotten from (6) and (3), via $\exists X\ (X = X)$ and $\emptyset = \{u : u \neq u\}$.
Abian (1969) proved consistency and independence of four of the Zermelo-Fraenkel axioms.

SEE ALSO: Axiom of Choice, Axiom of Extensionality, Axiom of Foundation, Axiom of Infinity, Axiom of the Power Set, Axiom of Replacement, Axiom of Subsets, Axiom of the Unordered Pair, Set Theory, von Neumann-Bernays-Gödel Set Theory, Zermelo-Fraenkel Set Theory, Zermelo Set Theory**REFERENCES:**

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Notes on the above version of ZFC are derived below from **Wikipedia** plus some comments and criticism by Dr. White. For more details see the full cited **Wikipedia** articles and related documents on the Internet.

In ZF "set" is a primitive notion, and "in ZF, everything is" [a set]. The axioms decide how these "sets" can interact. (**Wikipedia**, "Axiom of extensionality".)

Axiom #1 (Extensionality) basically says, "two sets are equal if and only if they have precisely the same members," which essentially means that "A set is determined uniquely by its members." An ur-element is a member of a set that is not itself a set. In the Zermelo–Fraenkel axioms, there are no ur-elements, but they are included in some alternative axiomatizations of set theory. Ur-elements can be treated as a different logical type from sets; in this case, $B \in A$ makes no sense if A is an ur-element, so the axiom of extensionality simply applies only to sets.(**Wikipedia**, "Axiom of extensionality".)

Axiom #2 (Pairing), states that if there exist two distinct sets, that there exists a set that contains exactly both of the two distinct sets. "Given two sets, there is a set whose members are exactly the two given sets.... Any two sets have a pair.... The axiom of pairing follows from the axiom schema of replacement applied to any given set with two or more elements...." (**Wikipedia**, "Axiom of pairing") This axiom introduces the notion of 2, and also the notion of iteration to generate sets containing ever larger groups of elements.

Axiom #3 (Subsets, Schema of Separation or Specification) states that "given any set *A*, there is a set *B* such that, given any set *x*, *x* is a member of *B* if and only if *x* is a member of *A* and φ holds for *x*.... There is one axiom for every such predicate φ; thus, this is an axiom schema.... Set *B* must be a subset of *A*.... Every subclass of a set that is defined by a predicate is itself a set. (**Wikipedia**, "Axiom schema of specification") "Given classes *A* and *B*, *A* is a subclass of *B* if and only if every member of *A* is also a member of *B*.[1] If *A* and *B* are sets, then of course *A* is also a subset of *B*. In fact, it's enough that *B* be a set; the axiom of specification essentially says that *A* must then also be a set. As with subsets, the empty set is a subclass of every class, and any class is a subclass of itself. But additionally, every class is a subclass of the class of all sets. Accordingly, the subclass relation makes the collection of all classes into a Boolean lattice, which the subset relation does not do for the collection of all sets. Instead, the collection of all sets is an ideal in the collection of all classes. (Of course, the collection of all classes is something larger than even a class!) (**Wikipedia**, "Subclass (set theory)") "The *axiom schema of comprehension* (unrestricted) reads...'There exists a set *B* whose members are precisely those objects that satisfy the predicate φ.' ...This axiom schema was tacitly used in the early days of naive set theory, before a strict axiomatization was adopted. Unfortunately, it leads directly to Russell's paradox by taking $\varphi(x)$ to be $\neg (x \in x)$ (i.e., the property that set *x* is not a member of itself). Therefore, no useful axiomatization of set theory can use unrestricted comprehension, at least not with classical logic." (**Wikipedia**, "Axiom Schema of Specification")

Axiom #4 (Axiom of the Sum Set) says that "for any set of sets *F* there is a set *A* containing every element that is a member of some member of *F*." (**Wikipedia**, "Axiom of union") An example is given. "The union over the elements of the set {{1, 2}, {2, 3}} is {1, 2, 3}."

Axiom #5 (Power Set) says, "For every set *x*, there is a set *P*(*x*) consisting precisely of the subsets of *x*.... Constructive set theory prefers a weaker version to resolve concerns about predicativity." (**Wikipedia**, "Axiom of power set")

Axiom #6 (Infinity) can be taken as the axiom of **existence** that asserts the existence of an undefined object to be called a set. This axiom also "guarantees the existence of at least one infinite set, namely a set containing the natural numbers" (**Wikipedia**, "Axiom of infinity"). "Infinity" really means that this undefined set can contain an undefined number of elements, each of which is different, ranging from none at all to as many as you like, an endless number of possible distinct elements. In that sense axiom 6 perhaps should be the first axiom. It establishes the possibility of an empty set along with any or all of the natural numbers as the possible cardinality of this undefined set. By allowing the notion of an empty set or a set with only one element, this set of axioms fundamentally departs from the "naive" concept of a set as a group of definite and distinct elements. Some versions of the ZF(C) axioms begin with the assertion of the existence of an empty set and build up from there. I believe that there can be no empty set, because a set is fundamentally the notion of embracing a group of elements -- which means two or more elements. The equivalent for 0 (the empty set) first occurs in the set of integers where 0 can effectively be represented by the set $\{+x, -x\}$, where *x* can be any natural number with the binary operators $(a + b)$ and $(a - b)$ employed. However, the notion of "set" as used in modern set theory is an abstract undefined metalinguistic primitive used for discussing relations among various elements, including relations with themselves and the absence of elements. The use of notions such as an empty set ∅ and 0 clearly leads to problems in mathematics as well as its physical interpretations and applications because of difficulties involved when nothing is allowed to interact with existential entities. Let us say that there is some finite or infinite set called Universe *U* that contains a subset *X* containing certain elements. The complement of *X* is the subset of all elements in *U* that are not members of *X*. We can call that complement *X′*, and it contains all the elements of *U* that are not members of *X*. Whatever set we take to be *U*, *U′* becomes the set of elements that are not in *U*, and that is the empty set ∅ relative to *U*. Also, ∅' = *U*. For example, the complement to the Universe of all 26 letters in the English alphabet is a set with no letters. The complement of the complement is the set of 26 letters. The complement to the set of **all** elements in any hypothetical universe *U* of elements therefore is an empty set. However, if we say that *U* contains **all possible** elements, then *U* cannot contain ∅, which by definition is not a true element. The complement to ∅ is all possible elements. My assumption here is that no set can be an element of itself and still be a set. It becomes an element. A set is a wholeness, and an element is a part. A wholeness cannot be a part of itself. (See axiom #8.) You cannot have a "Set of all sets," (or a set consisting of sets of any kind.) Therefore, you cannot have a "set of all sets that are not members of themselves." You can only have a set, which by its nature contains all its elements as members. Within a set you can have no sets, only elements. A "subset" is a group of elements within a set. A subset is not a set. A subset must be extracted from a set in order to become a set in its own right. This approach of distinguishing parts from wholes and language from metalanguage removes Russell's paradox by clarifying, but not sacrificing, our simple everyday understanding of what a set is.

Axiom #7 (Replacement) says that "There is a set **I** (the set which is postulated to be infinite), such that the empty set is in **I** and such that whenever any *x* is a member of **I**, the set formed by taking the union of *x* with its singleton {*x*} is also a member of **I**. Such a set is sometimes called an **inductive set**.... The other axioms [of ZFC] are insufficient to prove the existence of the set of all natural numbers. Therefore its existence is taken as an axiom—the axiom of infinity. This axiom asserts that there is a set **I** that contains 0 and is closed under the operation of taking the successor; that is, for each element of **I**, the successor of that element is also in **I**. Thus the essence of the axiom is: There is a set, **I**, that includes all the natural numbers.... The axiom of infinity cannot be derived from the rest of the axioms of ZFC, if these other axioms are consistent. Nor can it be refuted, if all of ZFC is consistent." (**Wikipedia**, "Axiom of infinity") See comments to Axiom #8 below.

Axiom #8 (Axiom of Regularity, also called Foundation) states that "every non-empty set *x* contains a member *y* such that *x* and *y* are disjoint sets. This implies, for example, that no set is an element of itself and that every set has an ordinal rank." (**Wikipedia**, "Zermelo-Fraenkel set theory") "A singleton is necessarily distinct from the element it contains, thus 1 and {1} are not the same thing, and the empty set is distinct from the set containing only the empty set.... A set is a singleton if and only if its cardinality is 1. In the standard set-theoretic construction of the natural numbers, the number 1 is ***defined*** as the singleton {0}." (**Wikipedia**, "Singleton (mathematics)". Regularity also implies that there is no infinite descending sequence of sets.

Axiom #9 (Choice) says that "given any collection of bins, each containing at least one object, it is possible to make a selection of exactly one object from each bin.... A choice function is a function *f*, defined on a collection *X* of nonempty sets, such that for every set

A in *X*, *f*(*A*) is an element of *A*. With this concept, the axiom can be stated: **Axiom** — For any set *X* of nonempty sets, there exists a choice function *f* defined on *X*." (**Wikipedia**, "Axiom of choice") This axiom involves making essentially simultaneous choice of one item from each of a finite or infinite collection of sets. The question of whether it is possible carry out such a choice function in the physical world brings up some interesting issues.

The existence of a multiplicity of equivalent axioms for ZF and ZFC suggests that there is no standard agreement on how to choose or arrange the axioms. (See the notes to the ZF axioms given at **Wikipedia**, "Zermelo-Fraenkel set theory", section 2.) Modern axiomatic set theory is based on emptiness, two undefined primitives linked by an undefined relation. It may be useful in many ways, but it has nothing to do with the original "naive" notion of a set, and I doubt that it is an appropriate "foundation" for mathematics.

White Set Theory

(Draft Version of a Different Approach to a More "Naive" Set Theory)

[I use a special vertical bar (|) notation to distinguish the relation between sets and their elements.]

Definition 1: Set signifies a group or collection of definite and distinguishable objects sharing certain similar properties that we will call elements, members, or elementary component parts of the set.

Definition 2: Group or **collection** signifies an undefined multiplex of elements. As members of a set, a group of elements shares some similar property.

Definition 3: Object signifies any mental or physical creation.

Definition 4: Creation signifies something defined (i.e., definite).

Definition 5: Distinguishable signifies that a creation can be perceived in some way as separate from another creation.

Definition 6: One signifies the fundamental wholeness of a set. We call it the essence of cardinality. The cardinality of a set with its total multiplex of elements treated as a whole subset is always 1|1. A set always has the cardinality of 1, because it represents a wholeness. The members (elements) of a set are the parts of the set's wholeness.

Definition 7: Two signifies the smallest group of elements distinguishable within a set: 1|2.

Definition 8: Property signifies a particular way of distinguishing elements.

Definition 9: A set *y* can be treated as a **subset** of a set *x* if and only if every element of *y* is also an element of *x*. Treated as a subset, y is no longer a set, but can be an element or consist of a group of elements, or even several subsubsets and/or subsets and elements.

Axiom 1: A set is determined by its members. (See ZF#1) Sets with the same multiplicity of elements are equal (=).

Axiom 2: A set is a whole and a member is a part, so a set may **not** be a member of itself.

Axiom 3: Sets may become parts of another set, but then the must be treated in that other set as a **subset** and demarcated with brackets []. Each such subset consists of all the elements of its source set. A set may not have a subset consisting of all its elements, for

then the part would no longer be a part, but would reflect the wholeness of the set.
Axiom 4: Any multiplicity of sets implies the existence of a larger set whose elements include all members of the multiplicity of sets.
Axiom 5: The smallest set is 1|2 (**one** wholeness consisting of **two** elementary parts).
Axiom 6: A smallest set or any multiplicity of smallest sets may become subsets of another smallest set, so long as the elements in the subsets are all definite and distinguishable from each other.
Axiom 7: The subsets within a set may be treated as elements or as containing subsubsets of any degree allowed by their multiplicity, each with a minimum of two elements. For example, a set consisting of two smallest subsets may also be treated as a smallest set of cardinality 1|2, or as a set containing a subset and two elements with cardinality 1|3, or as a group of elements with cardinality 1|4; (e.g., {[a, b], [c, d]}; {a, b, [c, d]}; {a, b, c, d}).
Axiom 8: For any set containing a multiplicity of subsets, the subsets may overlap in various ways depending on how the set as a whole is parsed into subsets and elements. Example: If set *A* = {[*a*, *b*], [*c*, *d*, *e*]}, it may also be parsed as {[*a*, *b*, *c*], [d, e]}. Set *A* in the example is parsed variously into two subsets, and element *c* is the **overlap** of the subsets as parsed in the two parsings given, but element *c* may not be duplicated simultaneously in two different subsets of a given parsing of a set, (e.g., {[*a*, *b, c*], [*c*, *d*, *e*]}. (Other parsings are possible for the above example. Duplications can be avoided by assigning subscripts to elements in different subsets if otherwise those elements would be identical.)
Axiom 9 (Infinity): There is a set ***I*** (the set which is postulated to be infinite), such that whenever any [*x*] is a subset of ***I***, the set formed by taking the union of subset [*x*] with the smallest group of elements [*a*, *b*] is also a member of **I**. Such a set is sometimes called an **inductive set**.... Example: {*a*, *b*}, {*a*, *b*, [x_a]}, {*a*, *b*, *c*, *d*}, {*a*, *b*, *c*, *d*, [x_b]}, {*a*, *b*, *c*, *d*, *e*, *f*}, ...
Axiom 11: The Axiom of the Power Set states that for any set *x*, there is a set *y* that contains every subset of x.
Theorem 1 (Well-ordering theorem) for Natural Numbers: For any set *X* there is a binary relation *R* which well-orders *X*. *R* is a linear order on *X* such that every subset of *X* has a member which is minimal under *R*.

There is no Axiom of Union in White's set theory as such, because such an axiom implies possible duplication of elements within various subsets, and that is not possible if every element of a set must be distinct in some way. The numbers +2 and -2 can be distinguished by the attaching of different operators to them. The numbers 2_a, 2_b, and so on, can be distinguished by subscripts and may exist simultaneously as members of the same or different subsets of a set.

Operational Postulates (Carried Out Only on Subsets and Elements within a Set)
Postulate 1 (Size): ($a < b$), ($a > b$), ($a = b$) hold for the set of natural numbers ℕ.
Postulate 2 (⊕): (a + b), (a - b) together with P1 hold for the set of integers ℤ.
Postulate 3 (⊗): (a × b), (a / b) together with P1 and P2 hold for the set of rationals ℚ.
Postulate 4 (ℂ): $(a)^x$, $(a^{(1/x)})$ together with P1, P2, and P3 hold for the sets of real, imaginary, and/or complex numbers (ℝ, 𝕀, ℂ). The imaginary set consists of multiples of $i = -1^{(1/2)}$. Minimal members of the complex set are all subsets consisting of the format (*a*, + *ib*) where *a* and *b* are real numbers. Real numbers in that set take the form

(a, + [ib, - ib]), and imaginary numbers in that set take the form ([a, - a], + ib).

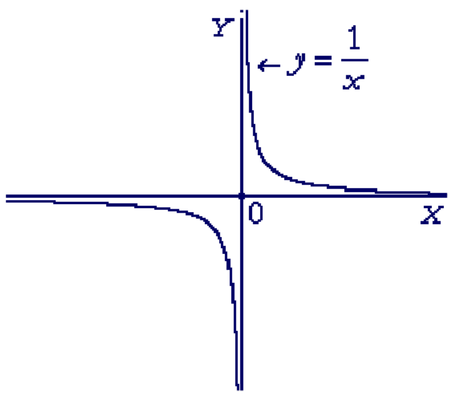
The set of natural numbers does not contain a "null set" with no members or a "singleton" set with one member, because a set by definition must contain at least two (2) members (cardinality (1|2)) forming a multiplex of elements (1|n), $n \geq 2$ that constitute the parts of the set. An "empty set" $S\varnothing$ is obtainable only from an operation on a set of integers. The simplest such null set: $S\varnothing \rightarrow \{+x, -x\}$, where x may be any element under P1 and P2. Here each element of $S\varnothing$ is distinct, and application of the operation of addition is carried out not on the members themselves, but their individual cardinal numeric values under the operation and results in the cancellation of the total "numeric value" of the members, but not of the members themselves that remain as distinct members of the set so that the set remains meaningful according to the definition of a set.

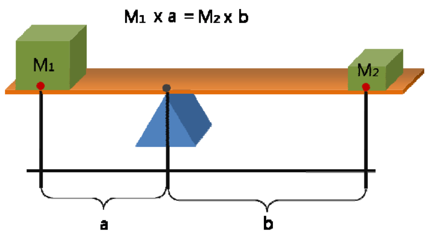
In a similar manner a set with a singleton (single positive resultant element) is obtainable by the integer operation of addition, and can be represented very simply as follows: $S1 \rightarrow \{[+a, +b], -a\}$. The operation is carried out on the elements and/or subsets as such, and affects only the "numeric value" attached to the elements or subsets, but not the elements and subsets themselves. For example, suppose $S1$ refers to a group of fruits consisting of an apple (+a) and a banana (+b). The apple and banana are placed on a table {+a, +b}, and then the apple is put away in a box: {[+a, -a], +b}. Both fruits still exist to compose the set, but only the single banana is on display. Removal of the apple cancels the presence of the apple on the table but not the existence of the apple. So we understand that the notion of "minus" (-) in the addition operation indicates that an element of a set is removed from consideration by the operation, but still "exists" as a member of the set. We can represent a singleton as the identity relation of addition: $S1 \rightarrow \{[+a, -a], +b\} \rightarrow \{[\varnothing], +b\} \rightarrow \{[1]\}$. The subset [$\varnothing$] still signifies [+$a$, -$a$], where a, unless specified, could be any element distinct and definite from other elements in the set. If element a is canceled from the set entirely leaving only the dummy subset [$\varnothing$], then information is lost regarding that set. $S1$ can also be generated from P4 in the following manner: $S1 \rightarrow \{a^{[+b, -b]}, [+c, -c]\}$, because any number raised to the 0 power is 1. The exponent [+b, -b] is a subset governed by the exponent operation, so that unitary set $S1$ contains altogether 5 elements: a, +b, - b, +c, -c. If internal details are not important, we can represent a singleton subset as [1] just as we have a null subset [$\varnothing$]. Any arbitrary symbol for a singleton subset [x] can be used in brackets since it represents a single element within a given set, other elements of the composite group in the set being "canceled" or otherwise present in other subsets.

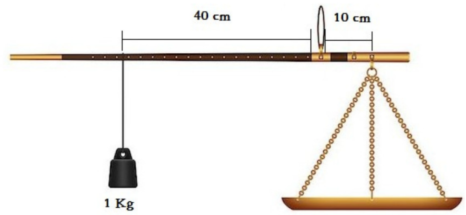
The identity operation with addition involves the null subset of elements: {[$\varnothing$], [+b]}. The identity operation with multiplication involves the singleton subset: {[1], $\times b$} or {b / [1]}. More generally, we can express it as $\{b \otimes [1]\}$.

In a case where there is the possibility of division by 0, we take the fundamental set to be: {[y, / [x, - x]], = a}. Here [y, / [x, - x]] is a subset and [x, - x] is a subsubset. We know that this must be the result of an operation on some function, $y = f(x)$. (For example, $y = 5x^2$.) We do not divide by 0, and we do not need to find a "limit". We reinterpret this as $\Delta y / \Delta f(x) = a$, and convert this to $dy/dx = anx^{(n-1)}$ based on the differential table derived by Mathis. In our example ($y = 5x^2$) we end up with $10x$ as the differential. In other words, we consider any case of apparent division by 0 as properly the derivative of a function and calculate the value of the differential as a constant value of a line that passes through the point in question that is tangent to the nonlinear function. In the case where $n = 0$, then the result of the division is also 0 and thus no problem. This takes care of a very large class of calculus functions. We should look briefly at three other common types of function: hyperbolic, trigonometric, and logarithmic just to see if we can really get along without avoiding division by "0" that requires leaping to a limit that transcends a series of infinitesimals in order to avoid an indeterminate result. It seems that 0 was brought into mathematics apparently just to be barred by an arbitrary law that mars the initial elegance of the number system and its algebraic operations. The presence of 0 headaches in mathematical calculations and in all sorts of numbering systems. (For more details, see **Wikipedia**, "0 (number)". The role of 0 in our notation perhaps should only be as a place holder, not as an actual numeric "value".

What if we have a hyperbolic equation such as ($y = 1 / x$)? We can rewrite this as ($x\ y = 1^2$) and supply proper units corresponding to the x and y dimensions. The ancients used a first-class lever or balance scale as a simple mechanical calculator for such equations. When the scale is balanced we have $m_a\ a = m_b\ b$, where the m's represent masses, while a and b are the displacements from the fulcrum on the sides where the subscript to m matches the letter for the displacement. The acceleration due to gravity is what approaches 0 as the scale comes into balance. At balance the accelerations in opposite directions cancel out to 0, and the ratios of the masses and displacements give the solution. There is no need to divide by 0. At the "limit" for acceleration the attention shifts from the acceleration of the balance to the ratio of displacement and mass. It is helpful to set up the scale so that you have a standard mass equaling 1 mass unit m and a standard displacement along an armature also equal to 1 spatial unit u. ($y = 1^2x^{-1}$); $dy/dx = nx^{n-1}$; $dy/dx = -(1^2/x^2)$.







The Chinese scale above on the right balances with a 4 kg weight in the pan at a displacement from fulcrum of 10 cm and the standard 1 kg weight at 40 cm: (1 kg)(40 cm) = (4kg)(10 cm). If 10 cm is taken as a standard unit of 1 u, and the standard unit weight is 1 m in kilograms, then ($x\ y\ [m/u]^2 = 1\ [m/u]^2$), where x is the ratio of a weight to be measured to the standard distance unit from the fulcrum, and y is the ratio of the standard weight to its distance from the fulcrum. In the above example we then have:

* $(x\ y) = (1m / 4u)\ (4m / 1u) = [1\ sm/su]^2$.

(Here sm is the standard mass, and su is the standard displacement unit; 4m is the test mass, and 4u, is the displacement of the standard mass required to reach the balance point.)

* $d\,(x^{-1})/dx = -1 / x^2$, (which is $-16\ (u/m)^2$ in our example.)

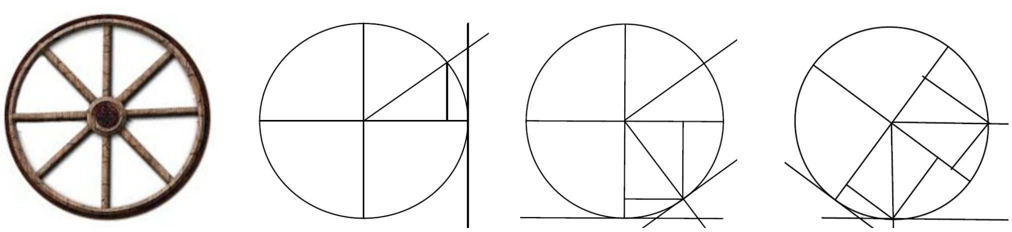
This hyperbolic equation establishes a simple reciprocal relation that gives the graph shown above in xy coordinates. For any given value the derivative is the slope of the tangent to the curve at the point corresponding to the given value. The standard weight m and standard displacement unit u create the gap that exists at the curve's closest "approach" to the origin point (0,0) on the graph of the equation, which is at point (1m, 1u) on the curve. We can calculate the Mathis differential for this equation: $-(1^2 / x^2)$. This gives us the constant slope ($-(1^2 / x^2) = -(4/1)^2$ for the straight line tangent. So when $x = 1/4$, the slope is a steep -16. If $x = 4$, the slope is a gradual -1/16. The minus sign means the slope is "downward" rather than "upward", regardless of whether x is positive or negative.

So what does all this have to do with 0, since the curve on the graph never goes there? The value of x tends **toward** 0 as y tends toward infinity, but the useful part of the curve never gets anywhere near 0 or infinity. As values of x or y approach 0 one would "scale" the scale to magnify the details. The true role of "0" occurs as the two arms of the scale oscillate over the fulcrum to find the balance point. When the acceleration of the weights in either direction goes to 0, the scale has found its balanced condition. But 0 is not relevant or necessary for the calculation of the balanced condition even though it is true that $F = mg$, where F is a force, m is a mass, and g is the acceleration of gravity, and also the two arms can balance. If the scale is out of balance, one side will fall faster than the other side. So we place a test weight in the pan and slide the standard weight until the two sides balance. Then we know exactly how much the test weight weighs. At the balance point the acceleration of the mass goes to 0 in both arms. The ancients have been doing a practical form of hyperbolic calculus for thousands of years (mechanical balancing scales are still used in many market places), and did not need Newton, much less Cauchy, Weierstrass or other members of the epsilon-delta gang trying to justify the notion of a "limit" by means of an infinitely decreasing displacement.

Next we take a look at the trigonometry functions that are widely used in both theoretical and practical mathematics. The standard differentiation of $f(x) = \sin x$ is $f'(x) = \cos x$. Then the usual process is to say that the limit as θ goes to 0 for $\cos \theta$ is 1, and the limit as θ goes to 0 for $\sin \theta$ is 0.

The trigonometry functions are ratios of the various pairs of sides of a right triangle with an acute angle. The simplest way to represent these is with a unit (radius = 1) circle: $x^2 + y^2 = 1$. With such a formula for any point on the circle the sine gives that point's y value above or below 0 and the cosine gives that point's x value above or below 0. A circle is a closed curve, and the derivative of a point on the curve should be the tangent line at that point on the curve. Each trig function is already a ratio of two straight lines that indicates a slope. On a unit circle the tangent (tan) indicates the tangent line relative to a particular angle and can be expressed as the ratio of sine to cosine. Or we can take the interval from the tangent point on, say, the x axis to where an extension of the radius at an angle to the x axis intercepts the tangent line relative to the unit radius. If the circle is sliding along that tangent line, then that is the "instantaneous velocity" at the tangent point. If the circle is rolling like a wheel along the tangent line, then it is the "instantaneous velocity" of the center of the circle. If the circle rolls at a constant rate, the velocity of the center is also constant.

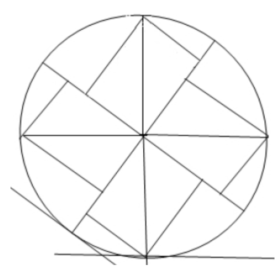
For the ancients, the mechanical version of the rolling circle is the wheel. A wheel on a wagon or a chariot rests on the ground, and flat ground represents a line tangent to the circle. If a line is extended from the axle out past the rim (to the right as we face the wheel) at any angle less than 90 degrees, then the tangent value is the distance from where the line touches the ground to where the rim touches the ground compared to the length of a spoke (axle center to edge of rim). If a peg is placed at the rim where the line crosses it, and the wheel is then rotated 90 degrees counter clockwise (until the vertical spoke points level to the right) and a weight is hung by a string from a peg on the rim, then the weight hangs down crossing the horizontal spoke at the point on it marking the sine along the string and the cosine along the spoke.



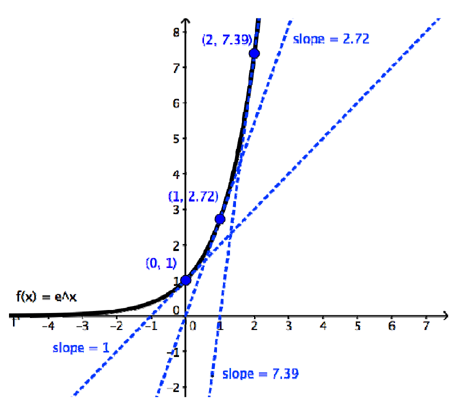
Suppose we place the wheel with the x axis touching the ground and then roll it to the right a distance of half a quarter of a turn. (See figure above on the right.) The center of the wheel moves forward parallel to the ground at a constant velocity as the wheel turns. The direction of center point motion is thus always parallel to the tangent line of the wheel touching the ground, no matter how far the wheel turns, and this tells us the velocity of the entire wheel as it rolls along the ground. As the wheel turns, the angle (we will call it *A*) made by the line from center to tangent point relative to the *x* axis changes, and sin *A* and cos *A* also change accordingly. Also the line through the center that is always parallel to the tangent line changes accordingly. The radius (*r*) from center to tangent point is always the diagonal of the rectangle formed by (sin *A*)(cos *A*), and *r* is set at unity. When the angle rotates an additional 90 degrees, the sine-cosine rectangle also rotates by 90 degrees into the next quadrant. Thus the sine and cosine swap places. The sine becomes the cosine, but the cosine becomes the sine with the sign changed since it is now on the other side of the center point (origin).

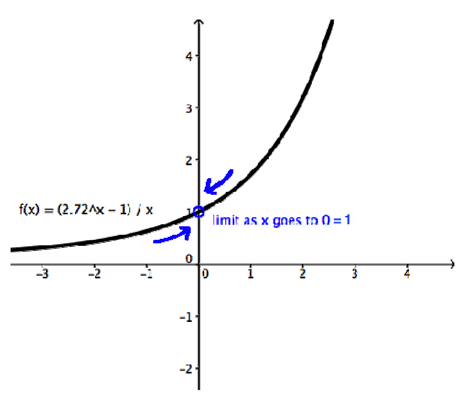
The addition rules of trigonometry confirm this. We will call the 90-degree shift angle *B*. We get sin (*A*+*B*) = sin*A* cos*B* + cos*A* sin*B*. The 90-degree value for cosine is 0 and the 90-degree value for sine is 1. So the result of the transformation is from sin *A* to cos *A*. Then also cos (*A*+*B*) = cos*A* cos*B* – sin*A* sin*B* = – sin*A*. This of course is what the derivatives of sine and cosine are. Thus we have the derivative for any value of the sine or cosine without going to 0 to find a limit, and we have a clear understanding of how it relates to a tangent point on a curve and a constant velocity. This is so much simpler than the otherwise necessary division of almost 0 by almost 0 using the "ghosts of departed quantities" as Berkeley so aptly described them.

The sketch below shows rotation by additional 90 degree steps into the third and fourth quadrant. The oscillating nature of sine and cosine are obvious.



Next we will take up the logarithmic functions exemplified by $y = e^x$. The symbol $e \approx$ 2.71828... represents the base of the natural logarithm, defined as the unique real number such that ln e = 1, that is, ($\log_e e^1 = 1$).





When mathematicians used the method of limits to find the derivative of $\boldsymbol{y = e^x}$, they found that $d\ (e^x)\ /\ dx = e^x$. To see if we can understand this situation more deeply without going to a proof by limits we will go back to our original goal: to **find the derivative of $\boldsymbol{y = e^x}$.** We can pose a hypothetical question:

What function would have a derivative that is the same as the function at any given point? This means that for every x value, the slope of the function's curve at that point is equal to the function's y value. What would be the function $y = f(x)$ on a number x such that it would be its own derivative $f\ '\ (x)$ -- and thus for every x value of the function the slope at that point would be the y value?

Here is the definition of a logarithm from **Wikipedia,** "Logarithm": **"**In mathematics, the **logarithm** is the inverse operation to exponentiation. That means the logarithm of a number is the exponent to which another fixed value, the base, must be raised to produce that number. . . . The logarithm of x to *base* b, denoted $\log_b(x)$, is the unique real number y such that $b^{\ y} = x$."

* $2^6 = 64$, then: $\log_2(64) = 6$

The **logarithm** of 2^2 to base 2 is 2, and the logarithm of 3^3 to base 3 is 3, the logarithm of 10^{10} to base 10 is 10. So we find that by the definition of logarithms the logarithm of n^n to **base $\boldsymbol{n}$** is the base n itself.

* $\log_n n^n = n.$ (by the definition of logarithm)

We can plug in any number. Even using base 0, then 0 to the 0^{th} power (0^0) would obviously seem to be 0, there being 0 numbers in base 0. This result is challenged by many who maintain that $0^0 = 1$ as it clearly is with all other positive numbers to the zeroth power. "In cardinal arithmetic, κ^0 is always 1 (even if κ is an infinite cardinal or zero)." (**Wikipedia**, "Exponentiation", section 8.) (e.g., $5^{(1-1)} = 5/5 = 1$.) Examples also are given in the evaluation of polynomials and in combinatorics where 0^0 must be evaluated as 1. However, in some of these cases this interpretation represents a symbolic unity at a different linguistic level. Another example given is $d\ x^n\ /\ dx = nx^{(n-1)}$, where $x = 0$ and $n = 1$. We get

* $d\ 0^1\ /\ d0 = (1)\ (0^{(1-1)}) = 0^1 / \ 0^1 = ?$

To me that "function" looks like $0^1 = 0$, which is reasonable, given that any number to the first power is that number itself. The "derivative" of such a hypothetical relation is 0/0, which appears to me indeterminate. However, $0^{(1-1)} = 0^0$, which suggests that $0/0 = 0^0$. But some prefer to force 0/0 to equal 1. This is just playing around with nothing, and gives meaningless mathematics, simply highlighting how the use of zero as a number in itself rather than just a place holder for calculation may not be mathematically sound reasoning. For a detailed discussion of the long running debate over how to interpret the zeroth power of zero see **Wikipedia**, "Exponentiation", section 10 "Zero to the power of zero".

The logarithm of 1^1 to base 1, is of course 1.

Now we hypothesize some number e to be the base for the equation $\log_n n^n = n$.. Then we let $n = e$.

* $\log_e \; e^e = e$. (An arbitrary substitution of e for n)

This looks like it could iterate endlessly: $\log_{e\wedge e} (e^e \wedge e^e) = e^e$.
(Here ^ means what follows is an exponent.)

Now we let the exponent part of our basic equation ($\log_e e^e = e$) become x. Thus,

(8) $\log_e (e^x) = x$. (definition of logarithm)

This function holds for all x, is considered continuous, and can be considered the definition of the natural logarithm (ln) which has the base e = 2.71828...; e.g.,

* $\log_e (e^4) = \ln (54.598...) = 4$.

Now we apply the "Mathis differential" to e^x to see what the derivative is.

* $e^x = x\, e^{x-1}$.
* $(1/x)\, e^x = e^x\, (1/e)$.
* $x = e$.

We cancel e^x and discover that in this situation $x = e$, which we of course knew. That means the derivative of e^x is e^x, and thus e^x always gives its own slope for the value of x. **However, this is also true for the derivative of <u>any</u> number n^n in the base n.** For example in our familiar base 10 notation: D $10^{10} = 10 \cdot 10^{10-1}$.= $10^{10-1+1} = 10^{10}$. And in general: D $n^n = n\, n^{n-1} = n^n$, because $\log_n n^n = n$ is so defined.

Thus we do not need to consider a limit to some quantity going to 0 in order to find the derivative of the natural logarithm. We know that e is irrational, but we can calculate an approximate value for it to any degree of precision we require, and we know that in base e the derivative of e^e will be $e\,(e^{e-1}) = e^{2e-1} = (1/e)\, e^{2e} = e^e$. We can substitute any number x for the exponent e and then $x = e$, and the derivative of e^x will be e^x.

The inverse of the function ($y = e^x$) becomes $y = \ln x$ or $e^y = x$. We also have an interesting reciprocal relation where $y = e^x$.

* $(D_x \ln x)\, (\ln e^x) = (1/x)\, (x/1) = 1$.

* $(D_x \ln x)(\ln y) = (1/x)(x/1) = 1$.

This reminds us of our hyperbolic equation $xy = 1$. Given ($y = e^x$), since $D_x \ln x = (1/x)$, we have that ($\ln e^x = x$) and ($y = D_x \ln x$); Thus, $xy = x/x = 1$.

The point of this detailed discussion of how to avoid arguments by "limit" processes is to show that the limit processes in calculus usually are intended to avoid the problem of division by 0. In the version of set theory that I propose there is no empty **set**. The appearance of an empty set *S*{}or ∅ in my suggested version of set theory, is a set consisting of a subset with two elements that mutually cancel under the operation of addition. Such a set appears only when we develop the set of integers with its possibility of cancellation within subsets, and this possibility extends into the rationals, reals, and complex numbers. This operation properly is performed on the elements of a set so as to remove a differential value between two numbers by subtracting an element from itself. In this chapter I have shown that it is not necessary to invoke limits that go toward or arrive at 0 in the calculus. In ordinary arithmetic we know that 0 only has meaning when the apples are sold out and we have to order some more. Otherwise we just remove them from our produce list and no longer carry them -- but they still exist. Thus 0 is meaningful only in terms of operational relations among existing **elements** of a set and does not exist as a **number** in its own right. We define numbers as symbols that indicate values, and 0 represents no value, so it cannot be a number in this view, only a potential relation among sets of numbers. In the future mathematics will be based on unity as it was in the ancient times before the "emptiness" people took over the shop. In Buddhism "emptiness" refers to the transitory nature of experiences in the realm of experience and transitory experiential existence is **not** nothing, but is a relation among various experiences.

From the above outline of a revived "naive" set theory we find that the **language of sets** is a **metalanguage**, and the **language of elements and the operations on them** is a **lower level language** (analogous to machine code and assembler language for computers as opposed to the programming languages that are metalanguages for the computer and closer to natural human language). In mathematics we often employ several levels of discourse simultaneously, so confusions can occur if we neglect to distinguish the various levels. The notation of subsets is a bridge that links the metalanguage of sets to the lower level language of elements, because a subset is a set that is treated as an element in some other set. The elements of a subset can be merged as elements with the elements of the set to which the subset belongs. Rules for algebraic and other operations on sets or the elements of sets also constitute metalinguistic grammars relative to the simple list of elements within a set and are not the same as the elements of a set.

B. A Glimpse at a Hierarchy of Sets that Extends into the Transfinite Realm

Standard set theory generates the natural numbers from an empty set symbolized by { } or ∅ plus an operation called "successor to ∅". "The **successor function** or **successor operation** is a primitive recursive function *S* such that $S(n) = n+1$ for each natural number *n*." (**Wikipedia**, "Successor function") This primitive function assumes unity and the operation of addition as primitive notions in order to define the natural numbers as emerging from an empty set ($n = 0$). The empty set is also an assumed primitive.

"In standard axiomatic set theory, by the principle of extensionality, two sets are equal if they have the same elements; therefore there can be only one set with no elements. Hence there is but one empty set, and we speak of "the empty set" rather than "an empty set"....

"For any set *A*: • The empty set is a subset of *A*. • The union of *A* with the empty set is *A*. • The intersection of A with the empty set is the empty set. • The Cartesian product of *A* and the empty set is the empty set.

"The empty set has the following properties: • Its only subset is the empty set itself. • The power set of the empty set is the set containing only the empty set. • Its number of elements (that is, its cardinality) is zero.

"The connection between the empty set and zero goes further, however: in the standard set-theoretic definition of natural numbers, we use sets to model the natural numbers. In this context, zero is modeled by the empty set.

"For any property: • For every element of ∅ the property holds (vacuous truth); • There is no element of ∅ for which the property holds. Conversely, if for some property and some set *V*, the following two statements hold: • For every element of *V* the property holds; • There is no element of *V* for which the property holds, $V = \varnothing$.

"By the definition of subset, the empty set is a subset of any set *A*. That is, every element x of ∅ belongs to *A*. Indeed, if it were not true that every element of ∅ is in *A* then there would be at least one element of ∅ that is not present in *A*. Since there are no elements of ∅ at all, there is no element of ∅ that is not in *A*. Any statement that begins "for every element of ∅ " is not making any substantive claim; it is a vacuous truth. This is often paraphrased as "everything is true of the elements of the empty set.

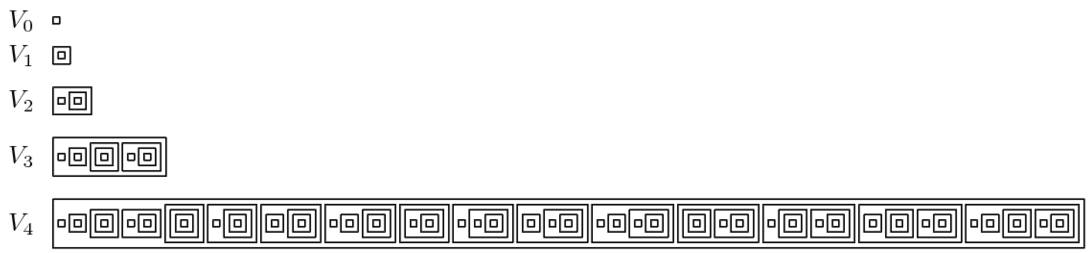
"Jonathan Lowe argues that while the empty set:

> "...was undoubtedly an important landmark in the history of mathematics, … we should not assume that its utility in calculation is dependent upon its actually denoting some object."

it is also the case that:

"All that we are ever informed about the empty set is that it (1) is a set, (2) has no members, and (3) is unique amongst sets in having no members. However, there are very many things that 'have no members', in the set-theoretical sense—namely, all non-sets. It is perfectly clear why these things have no members, for they are not sets. What is unclear is how there can be, uniquely amongst sets, a *set* which has no members. We cannot conjure such an entity into existence by mere stipulation." (**Wikipedia**, "Empty set".)

In the diagram of the von Neumann hierarchy of sets below V_0 represents the "empty set". However, in this presentation V_0 is an object that already plays multiple roles. It is itself. It is a set and is also its own element, so it is not really "empty". It is a "unity", and thus expresses the cardinality of 1 by combining the notions of set and element into a single entity so that the binary relation takes on the quality of an object referring to itself. (According to my interpretation of this, in V_0 we find 0, 1, and 2 already represented.)



"The set V_5 contains 2^{16}=65536 elements. The set V_6 contains 2^{65536} elements, which very substantially exceeds the number of atoms in the known universe. So the finite stages of the cumulative hierarchy cannot be written down explicitly after stage 5. The set V_ω has the same cardinality as ω. The set $V_{\omega+1}$ has the same cardinality as the set of real numbers....

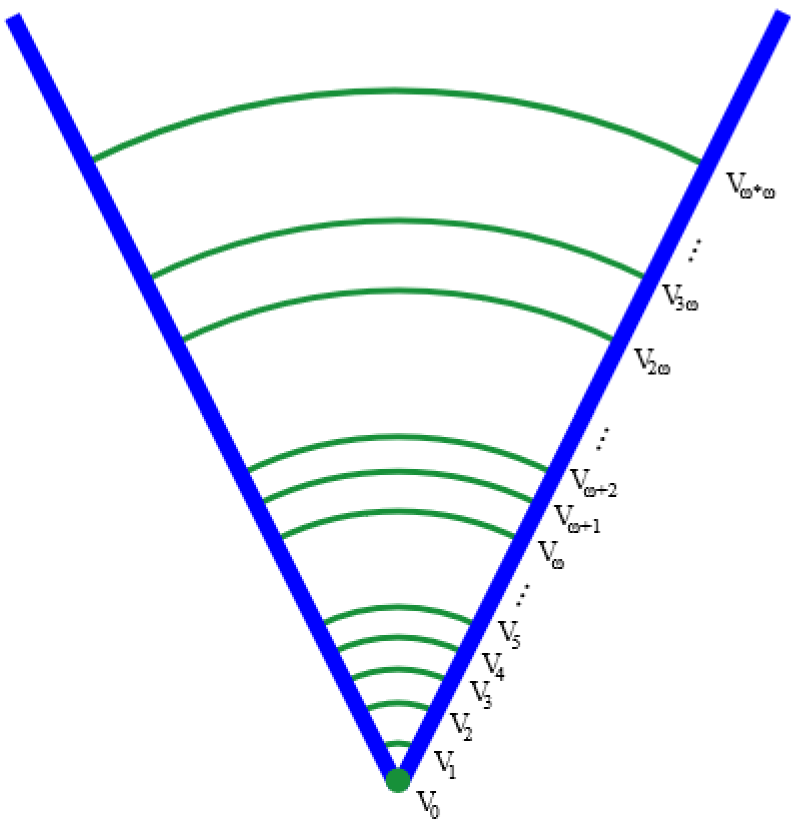
Since the class V may be considered to be the arena for most of mathematics, it is important to establish that it "exists" in some sense. Since existence is a difficult concept, one typically replaces the existence question with the consistency question, that is, whether the concept is free of contradictions. A major obstacle is posed by Gödel's incompleteness theorems, which effectively imply the impossibility of proving the consistency of ZF set theory in ZF set theory itself, provided that it is in fact consistent.

The integrity of the von Neumann universe depends fundamentally on the integrity of the ordinal numbers, which act as the rank parameter in the construction, and the integrity of transfinite induction, by which both the ordinal numbers and the von Neumann universe are constructed. The integrity of the ordinal number construction may be said to rest upon von Neumann's 1923 and 1928 papers. The integrity of the construction of V by transfinite induction may be said to have then been established in Zermelo's 1930 paper.

The cumulative type hierarchy, also known as the von Neumann universe, is claimed by Gregory H. Moore (1982) to be inaccurately attributed to von Neumann. The first publication of the von Neumann universe was by Ernst Zermelo in 1930.

Existence and uniqueness of the general transfinite recursive definition of sets was demonstrated in 1928 by von Neumann for both Zermelo-Fraenkel set theory and Neumann's own set theory (which later developed into NBG set theory). In neither of these papers did he apply his transfinite recursive method to construct the universe of all sets. The presentations of the von Neumann universe by Bernays and Mendelson both give credit to von Neumann for the transfinite induction construction method, although not for its application to the construction of the universe of ordinary sets.

The notation *V* is not a tribute to the name of von Neumann. It was used for the universe of sets in 1889 by Peano, the letter *V* signifying "Verum", which he used both as a logical symbol and to denote the class of all individuals. Peano's notation *V* was adopted also by Whitehead and Russell for the class of all sets in 1910. The *V* notation (for the class of all sets) was not used by von Neumann in his 1920s papers about ordinal numbers and transfinite induction. Paul Cohen explicitly attributes his use of the letter *V* (for the class of all sets) to a 1940 paper by Gödel, although Gödel most likely obtained the notation from earlier sources such as Whitehead and Russell."   (See **Wikipedia**, "Von Neumann universe".)



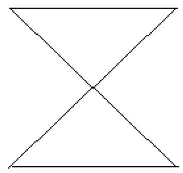
An initial segment of the von Neumann hierarchy of sets.   These sets are non-computable, and thus do not belong in applied mathematics. (See **Wikipedia**, "Set theory", section "Some Ontology".)   I call transfinite hierarchies of sets "**myth**ematics".

C. Diagram of the Interaction of World Space and Mind Space

World Space

Quantum Particles
(unpredictable, uncertain relations)
(discreteness)

Quantum Probability Waves
(certain, predictable relations)
(continuity)



Real Numbers
(unpredictable values,
uncertain relations)
(continuity)

Natural Numbers
(certain values, predictable relations)
(discreteness)

Mind (Math) Space

The crisscross structure of the above diagram has a quasi-fractal nature. As Mandelbrot, Wolfram, and others have pointed out, in the Mind Space it is possible to generate highly complex and perhaps even totally random results from very simple algorithms. It is also possible for a randomly initiated algorithm to collapse into a discrete, ordered system by means of an inherent attractor feature in its design. Likewise, in the World Space, objects that appear to be discrete particles tend to expand and dissolve their boundaries under the influence of entropy or wave packet destructive interference until they become continuous wave functions with no discernible particle nature. On the other hand, any World Space wave phenomenon tends to have an attractor such as gravity or an observer's measurement that can collapse it into one or more localized phenomena.

Therefore the use of mathematics in the study of physics may lead a student to realize how he has been confused by the deceptive reversals of predictability and certainty between Mind and World. A natural number exists forever in Mind Space. A natural object is here today and gone tomorrow in World Space. The flow of uncountable objects such as water, air, space, and consciousness reliably fills in gaps to give the World Space continuity and certainty. The "gap" numbers that fill in the Mental Space have no precise values. They are optional, randomly structured, and the observer must arbitrarily define their values. Yet there are crossover methods in each space and between each space. Attention to how these methods operate eliminates confusion.

There is an observer viewpoint in which a perfect match occurs between the Mind Space and the World Space. The World Space IS the Mind Space, and the Mind Space IS the World Space. This is sometimes called Unity Consciousness. We will consider such a possibility more formally in Chapter 6. In the meantime, explore the possibility.